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VERS LA THÉORIE DE L'INDICE DE CONLEY SUR LES COMPLEXES CUBIQUES

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Sommaire

Robin Forman a défini pour la première fois le champ vectoriel combinatoire sur les complexes simpliciaux en 1998. Kaczynski, Mrozek et Wanner ont proposé d'enrichir cette notion en flux combinatoire en 2016, ce qui permet de développer la théorie d'indice de Conley sur les complexes simpliciaux. Dans cette thèse, une preuve du chaos dans le modèle combinatoire de Lorenz par conjugaison est présentée. L'approche par flux combinatoire vers l'indice de Conley est adaptée aux complexes cubiques et la théorie est développée dans une certaine mesure. Un analogue cubique de l'application semi-continue supérieurement à valeurs contractibles est également établi.

Abstract

Combinatorial vector fields on simplicial complexes have been first defined by Robin Forman in 1998. The enrichment of this notion to combinatorial flow is proposed by Kaczynski, Mrozek, and Wanner in 2016, which allows the development of Conley index theory on simplicial complexes. In this thesis, a proof for chaos in combinatorial Lorenz model by means of conjugacy is presented. The combinatorial flow approach toward Conley index is adapted to cubical complexes and theory is developed to some extent. A cubical analogue of contractible valued upper semi continuous map is also established.

Introduction

In general, the study of continuous time dynamical systems can be a difficult task due the complexity of the flow behavior. A phase portrait of a dynamical system can be split into particular subsets called invariant, and the connecting orbits. An invariant set S has the property that an orbit initiating from S will always stay within S. In general, the study of invariant sets can be itself complex. A particular group of invariant sets called isolated invariant sets are of more interest in the study of dynamical systems as well as of isolating neighborhoods. Conley index in flow setting is an index associated with an isolating neighborhood of an isolated invariant set. This index is shown to be well-defined and independent in choice of isolating neighborhood for an isolated invariant set. The flow in some dynamical systems admits a Poincaré section which induces a Poincaré map. Study of the Poincaré map can retrieve information about the flow dynamic. Dynamics of maps can be studied by means of Conley index. The Conley index for flows and maps are defined and studied in [18]. In Chapter 1 Conley index for flows and maps and their properties are reviewed.

A combinatorial model for Lorenz system is suggested in [13]. The continuous time dynamic Lorenz system exhibits chaotic behavior that is proven in [17] and [19]. In Chapter 2 we study the dynamics of combinatorial Lorenz system. By means of a semi conjugacy between symbol space in two symbols and the combinatorial model for Lorenz system, the chaos on combinatorial Lorenz system is shown. Moreover, the Morse decomposition for combinatorial Lorenz system as well as its Morse connection graph are studied in Chapter 2. Then, a different model for combinatorial Lorenz system is suggested which differs in terms of dynamics from the classical Model. The Morse decomposition and Morse connection graph for the new model are also discussed. The minimality of the Morse decomposition for the Lorenz model and the modified Lorenz model are verified by use of Kosaraju's algorithm.

Combinatorial vector field is first defined on a simplicial complex by Robin Forman in [9]. The discrete Morse index of a combinatorial dynamical system is introduced and studied in [8]. Definition of a solution associated to a vector field on a simplicial complex given by Forman has two main flaws. First, with this definition, asymptotic behavior of an orbit in combinatorial setting is not possible and second, infinite time orbits do not exist in finite simplicial complexes. These two flaws were addressed in [13] by associating a flow to a combinatorial vector field. In the same source [13] multi-valued map approach is taken toward Conley index theory or a combinatorial dynamical system on a simplicial complex. There is a correspondence between the combinatorial solutions of the flow and the solutions of the multi valued map associated with a combinatorial vector field on a finite simplicial complex. In fundamental notions of that approach the barycentric coordinates play essential role. In Chapter 3 it is shown how that theory can be adapted to cubical complexes in absence of barycentric coordinates, by means of a certain Euclidean metric which will replace the role of barycentric coordinates. It is proven that the multi-valued map induced by a combinatorial vector field has non-empty, compact, and acyclic values and it is strongly upper semi continuous. One can find intense study on homology of cubical complexes in [16].

Chapter 1

Conley Index Theory on Continuous and Discrete Dynamical Systems

1.1 Invariant Set and Isolating Neighborhood

Let X denote a locally compact metric space. A continuous dynamical system defined on X can be represented by a differential equation or a system of differential equations. The set of solutions of such system forms a flow which has certain properties. Based on those properties the notion of flow on X can be defined independently as following.

Definition 1.1. *A map $\varphi : \mathbb{R} \times X \longrightarrow X$ represents a flow if φ is a continuous map satisfying*

(i) $\forall x \in X, \quad \varphi(0, x) = x$*,*

(ii) $\forall x \in X, \forall s, t \in \mathbb{R} \quad \varphi\big(t, \varphi(s, x)\big) = \varphi(t + s, x).$

The orbit passing through $x \in X$ is the set

$$\gamma_x = \{\varphi(t, x) \mid t \in \mathbb{R}\}. \tag{1.1}$$

If there is $T > 0$ such that $\forall t \in \mathbb{R},\ \varphi(t + T, x) = \varphi(t, x)$, then the orbit γ_x is called periodic.

In the study of dynamical systems, specially when the system is too complicated, one way to facilitate the study of the system behavior is to subdivide X into certain compact subsets so that an orbit which initiates from one of those subsets either stays within that subset or leaves the subset and arrive in another one. Inside each of these subsets which later will be called blocks, there might be a lot of dynamics going on which can itself be broken down into smaller blocks. Depending on the dynamical system study orientation and how expensive that could be, one may zoom into blocks to elaborate the studied system. A flow initiating from a block either stays in that block for positive and negative time or it leaves the block without coming back to it also it would not touch the boundary of block and come back to the block again, which is called internal tangency. Those criteria are formally defined as invariant sets and isolating invariant sets which are recalled later in this chapter.

Definition 1.2. *A set $S \subset X$ is an invariant set for the flow φ if*

$$\varphi(\mathbb{R}, S) := \bigcup_{t \in \mathbb{R}} \varphi(t, S) = S.$$

We note that, by first property of flow, it is obvious that $S \subseteq \bigcup_{t \in \mathbb{R}} \varphi(t, S)$. Therefore, in Definition 1.2, it is sufficient to check $\bigcup_{t \in \mathbb{R}} \varphi(t, S) \subseteq S$ is satisfied. In other words, a set S is called invariant if the flow initiating in S, stays in S in positive and negative time. A particular type of invariant set is a fixed point or critical point which flows initiating from that point will remain at that point for all positive and negative time.

Definition 1.3. *A point $x \in X$ such that $\varphi(\mathbb{R}, x) = \{x\}$, is called a fixed point or an equilibrium.*

An equilibrium point is a simple example of an invariant set. Proving the existence of fixed points for many differential equations, either analytically or numerically is not a trivial task. Degree theory is a standard tool for such type of problems.

The degree of a map can be thought of as a function from a set of continuous maps to the integers with three important properties:

(i) It is defined in terms of a region for which there are no fixed point on the boundary.

(ii) If the degree of a map is nonzero, then the map possesses a fixed point.

(iii) Degree is a continuous function.

Direct computation of degree of a map is often quite difficult. Therefore, the degree theory involves the following steps in order to find the degree of a map of interest.

- Choosing a continuous family of maps going from the map of interest to a simpler map for which the fixed points are explicitly known.
- Choosing a region containing one or more equilibria of the simple map for which the resulting degree is nonzero.
- Showing that each function in the continuous family of maps does not possess a fixed point on the boundary of the region.

Once these steps are passed through, the fact that the degree is integer valued and continuous, it can not change throughout the continuous family of maps, implies that the degree of the original map of interest is the same as that of the simpler map. In particular, the degree is nonzero and hence the original map has a fixed point. What makes this approach powerful, is that one does not need to study behavior of the map in the interior of the region.

The Conley index typically involves a similar process. The regions of interest in this theory are called isolating neighborhoods which are defined next.

Definition 1.4. *A compact set $N \subset X$ is an isolating neighborhood if*

$$\textit{Inv}(N, \varphi) := \{x \in N \mid \varphi(\mathbb{R}, x) \subset N\} \subseteq \textit{int}(N)$$

where int(N) means the topological interior of N.
The set S is an isolated invariant set if $S = \textit{Inv}(N)$ for some isolating neighborhood N.

The most important property of an isolating neighborhood is that it is robust with respect to perturbations. That means for a continuous family of dynamical systems, if N is an isolating neighborhood for one of the flows in that family say φ, then N is an isolating neighborhood for the flows in that family which are sufficiently close to φ. Precise definition of this is as follows.

A continuous family of dynamical systems on X is a set $\{\varphi_\lambda : \mathbb{R} \times X \to X\}_{\lambda \in I}$ of flows on X, where I is closed bounded interval, and the map $\phi(\lambda) = \varphi_\lambda$ defines continuous function from I into set of flows on X. The topology on the set $\{\varphi_\lambda : \mathbb{R} \times X \to X\}_{\lambda \in I}$ of flows on X is the compact-open topology.

Proposition 1.5. *Let $\{\varphi_\lambda\}_{\lambda\in[-1,1]}$ be a continuous family of flows on X, and N be an isolating neighborhood for the flow φ_0. Then, for sufficiently small $\delta > 0$, N is isolating neighborhood for all φ_λ, $|\lambda| < \delta$.*

Proof. One can find the proof in [18] □

The robustness of isolating neighborhoods leads to the following definition.

Definition 1.6. *Let $N \subset X$ be a compact set. Let $S_\lambda = Inv(N, \varphi_\lambda)$. Two isolated invariant sets S_{λ_0} and S_{λ_1} are related by continuation or S_{λ_0} continues to S_{λ_1} if N is an isolating neighborhood for all φ_λ, $\lambda \in [\lambda_0, \lambda_1]$.*

Observe that the definition of continuation is about isolating neighborhoods. It says nothing about the associated isolated invariant set.

What we are interested in is the structure and properties of invariant sets not isolating neighborhoods. Therefore we need a means by which we can pass from knowledge of isolating neighborhoods to an understanding of their associated invariant sets. The tool developed for this purpose is the Conley index theory.

1.2 Conley Index Theory of Flows and Maps

The Conley index for flow associated with a dynamical system was introduced and studied in [18]. In particular cases, the flow admits a Poincaré section and induces a Poincaré map which contains information about the flow. The Conley index for maps is defined and studied in [18]. In this section, definition of Conley index for flows and maps as well as their properties are reviewed.

The discussion of the Conley index has been restricted to dynamical systems generated by flows. However, the index theory can be extended to the maps setting. This extension is important even in context of differential equations. First, there are many situations in which the flow admits a Poincaré section and one can apply index theory to the associated Poincaré map. In this case the information retrieved by Conley index of the Poincaré map is essentially more than the information which is carried by index of the flow. The second reason is that numerical approximation of a flow take the form of a map.

The discrete dynamical systems generated by a homeomorphism $f : X \to X$ can be considered as the least technical system to be studied. Such homeomorphisms could be naturally constructed out of a given flow $\varphi : \mathbb{R} \times X \to X$. The time τ map, $\varphi_\tau : X \to X$ associated with φ is defined by $\varphi_\tau(x) = \varphi(\tau, x)$. It is straightforward to show that the map φ_τ is homotopic to identity map on X.

The ideas in Conley index theory of flows can be carried on for the discrete case, the dynamical system generated by a map. The notion of isolating neighborhood, the fundamental ingredient of Conley index theory in discrete case is given as following. Given $N \subseteq X$, the *maximal invariant set* of N is defined by

$$\text{Inv}(N, f) := \{x \in N \mid f^n(x) \in N \text{ for all } n \in \mathbb{Z}\}$$

A compact set N is called an *isolating neighborhood* if Inv$(N, f) \subseteq$ int N and a set S is an *isolated invariant* set if there exists an isolating neighborhood N such that $S =$ Inv(N, f).

Another notion which is carried from continuous case to discrete case is decomposition of isolated invariant sets. The results for flow case stay valid when the flow is replaced by a homeomorphism. The main difference between the setting of flows and maps (homeomorphisms) shows up in the definition of index. Conley index for maps is defined in [18] as a true generalization of Conley index for flows. If $S \subseteq X$ is an isolated invariant set for a flow φ, then S remains isolated invariant set for the time τ map associated with φ. Moreover, the Conley index associated with S in flow case and time τ map case agree.

1.2.1 Conley Index for Flows

In this section, the definition of Conley index for flows and its properties are reviewed.

A *pointed space* (Y, y_0) is a topological space Y with a distinguished point $y_0 \in Y$. For a pair of spaces N and L with $L \subseteq N$, the quotient space N/L is defined by an equivalence relation on N given by

$$\forall\, n_1, n_2 \in N,\;\; n_1 \sim_L n_2 \;\text{ if and only if }\; n_1 = n_2 \;\text{ or }\; n_1, n_2 \in L$$

Then

$$N/L := N/\sim_L$$

Which is the set of all equivalence classes of N with respect to relation $\sim_L$. The *quotient space* N/L is usually looked at as a pointed space $(N/L, [L])$. If N in the preceding definition is a topological space then so is N/L. The topology defined on N/L is given as follows: A set $U \subseteq N/L$ is open if U is open in N and $U \cap L = \emptyset$, or the set $\big(U \cap (N\backslash L)\big) \cup L$ is open in N.

Let (X, x_0) and (Y, y_0) be pointed topological spaces and let

$$f, g : (X, x_0) \to (Y, y_0)$$

be continuous functions of pointed spaces, i.e. $f(x_0) = g(x_0) = y_0$. f is called homotopic to g and denoted by $f \sim g$ if there exits a continuous function $F : X \times [0, 1] \to Y$ such that

$$\begin{aligned} F(x, 0) &= f(x), \\ F(x, 1) &= g(x), \\ F(x_0, t) &= y_0, \ \forall t \in [0, 1]. \end{aligned}$$

Homotopy $\sim$ defines an equivalence relation on the set of continuous functions from (X, x_0) into (Y, y_0). The equivalence class of f is denoted by $[f]$.

Two pointed spaces (X, x_0) and (Y, y_0) are called homotopic and it is denoted by $(X, x_0) \sim (Y, y_0)$, if there exist continuous maps $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (X, x_0)$ such that

$$f \circ g \sim id_Y \ \text{ and } \ g \circ f \sim id_X.$$

Homotopy defines an equivalence relation on the set of pointed topological spaces.

We now have recalled the required notions in order to define the Conley index for flows. Assume X is a locally compact metric space and φ is a flow on X.

Definition 1.7. *Let S be an isolated invariant set. An index pair for S is a pair of compact sets (N, L) such that*

(i) $S = Inv(cl(N\backslash L))$ *and* $N\backslash L$ *is a neighborhood of* S.

(ii) L *is positively invariant in* N, *i.e. given* $x \in L$ *and* $\varphi([0, t], x) \subset N$, *then* $\varphi([0, t], x) \subset L$

(iii) L *is an exit set for* N, *i.e given* $x \in N$ *and* $t_1 > 0$ *such that* $\varphi(t_1, x) \notin N$, *then there exists* $t_0 \in [0, t_1]$ *for which* $\varphi([0, t_0], x) \subset N$ *and* $\varphi(t_0, x) \in L$.

Second condition prevents the flows originating from $N\backslash L$ to touch L and come back to $N\backslash L$. The third condition implies that flows can leave N only through the set L.

Here we bring some results for Conley index of flows from [18] without proof in order to get a better intuition about this concept.

For a given isolated invariant set S, the index pair always exists. More precisely, there exists a compact set B such that $S = \text{Inv}(B)$ and (B, B^-) is an index pair for S where

$$B^- = \{x \in B \mid \forall_{t>0},\ \varphi([0,t), x) \not\subset B\}$$

is closed, and

$$\forall_{t>0},\ \text{Inv}_t(B, \varphi) = \{x \in B \mid \varphi([-t,t], x)\} \subset \text{int}(B).$$

The set B above is called an *isolating block*.

The homotopy Conley index for an isolated invariant set S in terms of an index pair (N, L) for S is defined as

$$h(S) := (N/L, [L])$$

The homotopy Conley index is the homotopy type of topological spaces which makes it extremely difficult to find Conley index through this definition. To resolve this obstacle for index calculation, the homological Conley index is defined as

$$\text{CH}_*(S) := H_*(N/L, [L]).$$

Since Conley index is defined for any index pair, one needs to prove that Conley index is well defined. That is if (N, L) and (N', L') are both index pairs for an isolated invariant set then the Conley index in terms of (N, L) is the same as Conley index in terms of (N', L'). It is shown that for two index pairs (N, L) and (N', L') for an isolated invariant set, the pointed topological space $(N/L, [L])$ is homtopic to the pointed topological space $(N'/L', [L'])$. This proves that the homotopy Conley index is well defined. Moreover, the homotopic spaces have the same homology, consequently, the homology Conley index is well defined as well.

Conley index possesses the continuation property which means that, for isolated invariant sets S_{λ_0} and S_{λ_1}, the Conley index of S_{λ_0} and Conley index of S_{λ_1} are identical.

Another property of Conley index, called *Ważewski property* states that for an isolating neighborhood N,

$$\text{CH}_*(\text{Inv}(N)) \ncong 0 \Longrightarrow \text{Inv}(N) \neq \emptyset.$$

Finally, the *summation property* of Conley index is stated as follows: assume $S = S_0 \cup S_1$ is an isolated invariant set where S_0 and S_1 are disjoint invariant sets, then

$$\text{CH}_*(S) \cong \text{CH}_*(S_0) \oplus \text{CH}_*(S_1).$$

Conley index characteristics can be summarized as following :

(i) The Conley index is an index of isolating neighborhoods. Furthermore, if N and N' are isolating neighborhoods for flow φ and $\text{Inv}(N, \varphi) = \text{Inv}(N', \varphi)$, then the Conley index of N is the same as the Conley index of N'. Therefore, one can consider the Conley index as an index of isolated invariant sets.

(ii) (*Ważewski Property*) If the Conley index of N is not trivial, then $\text{Inv}(N) \neq \emptyset$.

(iii) (*Continuation*) If N is an isolating neighborhood for a continuous parameterized family of flows φ_λ, $\lambda \in [0, 1]$, that is

$$\text{Inv}(N, \varphi_\lambda) \subset \text{int}(N), \text{ for } \lambda \in [0, 1]$$

then the Conley index of N under φ_0 is the same as the Conley index of N under φ_1.

The first property of Conley index gives great freedom in choice of regions in phase space on which one will perform the analysis. The second allows one to pass from the isolating neighborhood to an understanding of the dynamics of the isolated invariant set. While the Ważewski property is the most fundamental property which gives general information about something non-trivial happening inside the isolating neighborhood, there are more sophisticated theorems that guarantee existence of connecting orbits, periodic orbits, and chaotic dynamics in the sense of symbolic dynamic. Because of a large amount of preliminaries required we skip to review those theorems. They can be found in section 4 of [18]. The importance of the third property is coming from the fact that the Conley index is a purely topological index and as such it is a very coarse measure of dynamics.

1.2.2 Conley Index for Maps

In this section we will see the definition of Conley index extended to Discrete dynamical systems, that is, the Conley index for a continuous map $f : X \to X$. Same as the case of flow, the Conley index for discrete dynamical system is defined based on index pairs. Index pair for a discrete dynamical system is defined next.

Definition 1.8. *Let X be a locally compact metric space and let $S \subseteq X$ be an isolated invariant set. The pair (N, L) of compact subsets of X with $L \subset N$ is called an index pair for S if*

(i) $cl(N \backslash L)$ is an isolating neighborhood for S,

(ii) L is positively invariant w.r.t. map f, i.e. $f(L) \cap N \subset L$,

(iii) L is exit set of N w.r.t. map f, i.e. $f(N \backslash L) \subset N$.

It is shown in [18] that for any neighborhood V of an isolated invariant set S, there exists an index pair (N, L) for S such that $N \backslash L \subset V$.

Generalizing the homotopy Conley index from flow case to case of maps fails to work out because for two index pairs (N, L) and (N', L') of an isolated invariant set S, the quotient spaces $N/ \sim_L$ and $N'/ \sim_{L'}$ are not necessarily homotopic. There was not such problem in flow case as the homotopy between those two quotient spaces could be constructed along the trajectories of flows which is not available any more in discrete dynamical systems. Therefore, homotopy Conley index for flows can not be used for case of maps as Conley index would not be well-defined anymore. To fix such inconvenience, for an index pair (N, L) of an isolated invariant set S w.r.t map f, the associated *index map* is the map $f_{N,L} : N/L \to N/L$ defined by

$$f_{N,L}([x]) = \begin{cases} f(x) & \text{if } f(x) \in N, \\ [L] & \text{otherwise.} \end{cases}$$

Then, the map $f_{N,L}$ is continuous. Moreover, for index pairs (N, L) and (N', L') of an isolated invariant set S, the homotopy classes of index maps $[f_{N,L}]$ and $[f_{N',L'}]$ are shift equivalent. This means there exist continuous maps $r : N/L \to N'/L'$ and $s : N'/L' \to N/L$ and a natural number m such that $r \circ f_{N,L} \sim f_{N',L'} \circ r$, $f_{N,L} \circ s \sim s \circ f_{N',L'}$, $r \circ s \sim {f_{N,L}}^m$, and $s \circ r \sim {f_{N',L'}}^m$.

The homotopy Conley index for an isolated invariant set S is defined as

$$h(S, f) := [f_{N,L}]$$

where (N, L) is any index pair for S and $[f_{N,L}]$ is the shift equivalence class of the index map $[f_{N,L}]$.

By this definition, the homotopy Conley index for maps is well defined. It also possesses the same basic properties as the Conley index for flows such as Wȧzewski property, continuation property, and summation property. But homotopy Conley index is difficult to compute. Therefore, as in the case of flows, the alternative definition of Conley index for maps is given in terms of homology. The *homology Conley index* for a map f and an isolated invariant set S is denoted by $Con_*(S, f)$ and is defined as the shift equivalence of $\mathrm{H}_*(f_{N,L})$.

1.3 Morse Decomposition

In many situations the dynamical system behavior is very complicated, thus as way to reduce the complexity of the system and facilitating the system behavior study, one can decompose the sets into specific invariant subsets. In this section we will see the coarsest of such decomposition as attractor-repeller decomposition which leads to possibly finer decomposition and generalized notion named as Morse decomposition.

Let X a locally compact metric space and let $x \in X$ and $\varphi : \mathbb{R} \times X \to X$ a flow on X. The *alpha* and *omega limit* sets of x under φ are defined respectively by

$$\alpha(x, \varphi) = \bigcap_{t>0} \mathrm{cl}\big(\varphi\big((-\infty, -t], x\big)\big) \text{ and } \omega(x, \varphi) = \bigcap_{t>0} \mathrm{cl}\big(\varphi\big([t, \infty), x\big)\big).$$

If X is a compact set, then for any $x \in X$ and flow φ on X, the sets $\alpha(x, \varphi)$ and $\omega(x, \varphi)$ are non-empty, compact and invariant.

Development of above notions for a set $Y \subseteq X$ and a flow $\varphi : \mathbb{R} \times X \to X$ are given as the *alpha* and *omega limit sets* of Y under flow φ and defined respectively as

$$\alpha(Y, \varphi) = \bigcap_{t>0} \mathrm{cl}\big(\varphi((-\infty, -t], Y)\big) \text{ and } \omega(Y, \varphi) = \bigcap_{t>0} \mathrm{cl}\big(\varphi([t, \infty), Y)\big).$$

For a compact invariant set S, the subset $A \subseteq S$ is called an *attractor* in S if for some neighborhood U of A,

$$\omega(U \cap S, \varphi) = A.$$

The *dual repeller* of A in S is defined by

$$R := \{x \in S \mid \omega(x, \varphi) \cap A = \emptyset\}.$$

The pair (A, R) is called and *attractor-repeller pair decomposition* of S.

Let S be an isolated invariant set and (A, R) be an attractor-repeller decomposition pair for S. Then A and R are both isolated invariant sets. Moreover, if A' is an attractor in A, then A' is an attractor in S. Next, by definition of the sets A and R, there is no orbit from attractor to repeller but there might be ones from repeller to attractor. Therefore, the set of connecting orbits for the attractor-repeller decomposition pair (A, R) for S, is defined by

$$\mathrm{C}(R, A; S) = \{x \in S \mid \alpha(x, \varphi) \subset R,\ \omega(x, \varphi) \subset A\}.$$

It can be shown that for a compact invariant set S, we have

$$S = R \cup A \cup C(R, A; S).$$

The attractor-repeller decomposition for a compact invariant S is the coarsest decomposition for S. Preceding argument results in the following definition as a generalization of attractor-repeller decomposition.

A set Λ with a relation $>$ is called *strict partially ordered* if

(i) $\forall a \in \Lambda,\ a \not> a$,

(ii) $\forall a, b, c \in \Lambda, (a > b,\ b > c \to a > c)$.

If in addition the partial order satisfies

(iii) $\forall a, b \in \Lambda,$ either $a > b$ or $b > a$,

then $>$ is called a *total order*.

Definition 1.9. *Let S be a compact invariant set. A finite collection of disjoint compact, invariant subsets of S, $\{M_\lambda \mid \lambda \in \Lambda\}$ is called a Morse decomposition if there is a partial order $>$ on the index set Λ such that for every $x \in S$,*

$$x \in S \setminus \bigcup_{\lambda \in \Lambda} M_\lambda \longrightarrow \exists\, a, b \in \Lambda, a > b,\ \alpha(x, \varphi) \subset M_a \text{ and } \omega(x, \varphi) \subset M_b.$$

The sets M_λ in Definition 1.9 are called *Morse sets*. Any partial order on Λ for which $\{M_\lambda \mid \lambda \in \Lambda\}$ is a Morse decomposition of S is called *admissible*.

In the next chapter we will see the notion of Morse decomposition in combinatorial setting. Then we exhibit such decomposition for some combinatorial dynamical systems.

Chapter 2

Study of Combinatorial Lorenz Model

The time continuous *Lorenz* system is given by the following system of differential equations:

$$\begin{cases} \dot{x} = s(y - x) \\ \dot{y} = x(R - z) - y \\ \dot{z} = xy - qz \end{cases} \tag{2.1}$$

where (s, R, q) are the system parameter values. In [17] by means of computer assisted computations, it is proven that, for certain open sets of parameters, the system exhibits chaotic behavior. The detailed proof to this for a sufficiently small neighborhood of system parameter values $(s, R, q) = (45, 54, 10)$ is given in [19]. The proof is based on showing the existence of a Poincaré section N for the section $P = \{(x, y, z) \mid z = 53\}$. Next, a conjugacy between a maximal invariant set of Poincaré section, $\text{Inv}(N, g) := \{x \in N \mid \forall\ n \in \mathbb{Z}, g^n(x) \in N\}$, and symbol space on two symbols Σ_2 is shown, where g is the associated Poincaré map with N induced by Equation 2.1. More precisely, it is shown that there exists $d \in \mathbb{N}$ and a continuous surjection $\rho : \text{Inv}(N, g) \to \Sigma_2$ such that $\rho \circ g^d = \sigma \circ \rho$ where $\sigma : \Sigma_2 \to \Sigma_2$ is the full shift dynamics on two symbols. Consequently, one can conclude that for some iterates the dynamic of Poincaré map g is at least as complicated as the dynamic of symbol space on two symbols.

A combinatorial model for Lorenz system in simplicial complex setting is given in [13]. In this chapter we aim to show the chaos on this combinatorial dynamical system similar to how it is done for continuous time Lorenz system by means of a conjugacy map between the two dynamical systems.

2.1 Symbol Space on n Symbols

Let X be a locally compact metric space and let $\varphi : \mathbb{R} \times X \to X$ be a flow on X. The time τ map, $\varphi_\tau : X \to X$ given by $\varphi_\tau(x) = \varphi(x, \tau)$ is continuous and also homotopic to the identity map on X. The notion of map on a locally compact metric space can be also defined independently and then its time discrete dynamic be studied. To avoid complexity, the maps of study are assumed to be a homeomorphisms. In this case, we may not necessarily have homotopy equivalence to identity map anymore. Studies on dynamics generated by a map can be found in [18], [16], [21], [22], [28], [23], [24].

For a map $f : X \to X$ and $x \in X$, the *full orbit* or *full solution* of x under f is the sequence $\{x_n\}_{n\in\mathbb{Z}}$ in X such that

$$x_0 = x, \text{ and } \forall_{n\in\mathbb{Z}}\ x_{n+1} = f(x_n). \tag{2.2}$$

The subsequence $\{x_n\}_{n\geq 0}$ of $\{x_n\}_{n\in\mathbb{Z}}$ is called the *forward orbit* under f through x. The forward orbit through x can be also written as

$$\gamma^+(x, f) := \{f^n(x) \mid n = 0, 1, 2, \cdots\}. \tag{2.3}$$

The forward orbit of $x \in X$ is a subset of X and it is easy to observe that $f(\gamma^+(x, f)) \subset \gamma^+(x, f)$. In general the last inclusion can remain proper.

The subsequence $\{x_n\}_{n\leq 0}$ of $\{x_n\}_{n\in\mathbb{Z}}$ is called a *backward orbit* of x under f. If f is onto, then a backward solution for every $x \in X$ exists but it is not necessarily unique, unless f is additionally one to one. If f is invertible then backward orbit of through x is unique and can be written as

$$\gamma^-(x, f) := \{f^{-n}(x) \mid n = 0, 1, 2, \cdots\}. \tag{2.4}$$

If f is an invertible map then for every $x \in X$, the forward orbit $\{x_n\}_{n\geq 0}$ of x under f can be uniquely extended to a full solution $\{x_n\}_{n\in\mathbb{Z}}$ through x by defining

$$x_{-n} = f^{-n}(x). \tag{2.5}$$

Definition 2.1. *Let $f : X \to X$ be a continuous map and let $x \in X$. A full solution through x under f is a function $\gamma_x : \mathbb{Z} \to X$ satisfying the following two properties:*

(i) $\gamma_x(0) = x$*,*

(ii) $\gamma_x(n+1) = f(\gamma_x(n))$ *for all* $n \in \mathbb{Z}$*.*

We note that if f is not invertible, extension of forward orbit to a full solution is not guaranteed, and even if such extension is possible, it may not be necessarily unique.

A subset $S \subset X$ is called *invariant set* under a continuous map $f : X \to X$, if $f(S) = S$. Having this definition, a forward orbit is not necessarily an invariant set, while if in addition f is invertible then $\gamma^-(x, f) \cup \gamma^+(x, f)$ is invariant under f.

One of the goals in the study of dynamical systems is to discover techniques and methods to understand the existence of invariant sets and their structure. Because of a tremendous variety of orbits type and combination of those which may occur, this goal will be extremely difficult to achieve in general. A particular discrete space, called the *symbol space* which is indeed a metric space, could be used as a tool to assess the behavior complexity of another continuous or discrete time dynamical systems by means of conjugacy or semi-conjugacy. The symbol space along with a certain map defined on it can be a model for a chaotic discrete dynamical system. The precise definition is given next.

The *symbol space* on n symbols is defined as

$$\begin{aligned}\sum\nolimits_n &= \{1, 2, \cdots, n\}^{\mathbb{Z}} \\ &= \{(\cdots, a_{-1}, a_0, a_1, \cdots) \mid a_i \in \{1, ..., n\} \text{ for all } i \in \mathbb{Z}\}.\end{aligned}$$

The space $\sum_n$ can be turned into a metric space by the metric d defined on $\sum_n$ to be

$$d(a, b) = \sum_{n=-\infty}^{\infty} \frac{\delta(a_n, b_n)}{4^{|n|}} \tag{2.6}$$

where,
$$\delta(s,t) = \begin{cases} 0 & \text{if } s = t, \\ 1 & \text{if } s \neq t. \end{cases}$$

The shift map on n symbols $\sigma : \sum_n \longrightarrow \sum_n$ is defined by

$$(\sigma(a))_k = a_{k+1}. \tag{2.7}$$

The map σ is a uniformly continuous function with respect to the given metric on $\sum_n$. Indeed σ is a homeomorphism and it fits to invertible systems. A subset of $\sum_n$ which fits to non-invertible dynamics is defined by

$$\begin{aligned} \sum\nolimits_n^+ &= \{1, 2, \cdots, n\}^{\mathbb{Z}^+} \\ &= \{a = (a_0, a_1, \cdots) \mid a_i \in \{1, ..., n\} \text{ for all } i \in \mathbb{Z}^+\}. \end{aligned} \tag{2.8}$$

The subset $\sum_n^+ \subset \sum_n$ can be endowed with a slightly different metric d' defined as following.

$$d'(a,b) = \sum_{n=0}^{\infty} \frac{\delta(a_n, b_n)}{2^n}. \tag{2.9}$$

The space $\sum_n^+$ can be considered as a subspace of $\sum_n$ by embedding, then, metrics d and d' defined on $\sum_n^+$ are equivalent.

2.2 Combinatorial Vector Field

The notion of vector field in combinatorial setting has been initially introduced by Robin Forman in [9]. This notion is followed by a vast amount of research and results in [8], [11], [10]. Later on, in [13], a slight modification in notion of combinatorial vector field suggested by Robin Forman is made. That smoothed the path toward developing the idea of isolated invariant sets and index pair in combinatorial case and therefore being able to associate Conley index to each finite combinatorial dynamical system. The combinatorial solutions under a combinatorial vector field V or a V-paths of index p defined by Forman, have a certain restriction that is, a V-path with index p can only contain simplices of dimension

p and $p + 1$. That is an inconvenience in demonstrating asymptotic behavior of an orbit in combincatorial setting. The development introduced for combinatorial orbits in [13] goes around the Forman's limitation and permits the dimension of simplices in an orbit to increase or decrease by even more than one. That is great as it allows us to have the notion of asymptotic orbits being transferred from continuous time setting to discrete time and so, the combinatorial system corresponding to a time continuous system can fit better into its background time continuous dynamics. Moreover, even on finite simplicial complexes such development in [13] admits full solutions and there are finite solutions which may be extended to full solutions.

We start with the notion of *Combinatorial Vector field* on a finite simplicial complex $\mathcal{K}$ given by Forman in [9].

Definition 2.2. *Let $\mathcal{K}$ be a finite simplicial complex. A combinatorial vector field on $\mathcal{K}$ is a map $V : \mathcal{K} \to \mathcal{K} \cup \{0\}$ such that*

(i) $\forall\ \sigma \in \mathcal{K}\ \Big(V(\sigma) \neq 0 \longrightarrow \mathit{dim}(V(\sigma)) = \mathit{dim}(\sigma) + 1,\ \mathit{and}\ \sigma \in \mathit{Bd}(V(\sigma))\Big)$,

(ii) $\forall\ \sigma, \tau \in \mathcal{K}\ \Big(V(\sigma) = \tau \neq 0 \longrightarrow V(\tau) = 0\Big)$,

(iii) $\forall\ \sigma, \tau \in \mathcal{K}\ \mathit{card}(V^{-1}(\sigma)) \leq 1$.

Where $\sigma \in \mathrm{Bd}(V(\sigma))$ means σ is in combinatorial boundary of $V(\sigma)$, together with $\dim(V(\sigma)) = \dim(\sigma) + 1$, it means σ is a facet of $V(\sigma)$. This definition looks a bit hard to read moreover the combinatorial vector field is defined from $\mathcal{K}$ to $\mathcal{K} \cup \{0\}$, where 0 is just an auxiliary symbol. Those simplices which are mapped to 0 and do not belong to image of V are called the *critical* simplices.

The modified version of a combinatorial vector field suggested in [13] is as follows:

Definition 2.3. *Let $\mathcal{K}$ denote a simplicial complex. A combinatorial or discrete vector field on $\mathcal{K}$ is an injective partial function $\mathcal{V} : \mathcal{K} \nrightarrow \mathcal{K}$ which satisfies the following conditions:*

(i) $\forall\ \sigma \in \mathcal{K}\ \Big(\mathcal{V}(\sigma) = \sigma\ \mathit{or}\ \{\sigma \in \mathit{Bd}(\mathcal{V}(\sigma))\ \mathit{and}\ \mathit{dim}(\mathcal{V}(\sigma)) = \mathit{dim}(\sigma) + 1\}\Big)$,

(ii) $\mathcal{K} = \mathit{Dom}(\mathcal{V}) \cup \mathit{Im}(\mathcal{V})$,

(iii) *Dom*$(\mathcal{V}) \cap$ *Im*$(\mathcal{V}) =$ *Fix*$(\mathcal{V})$.

The first condition states that image of a simplex is either itself or it is one of its facets. The second condition states that elements of $\mathcal{K}$ are either in Dom$(\mathcal{V})$, or in Im$(\mathcal{V})$, or in both. Finally, the third condition states those simplices which are both in Dom$(\mathcal{V})$, and in Im$(\mathcal{V})$ are *fixed* or *critical* simplices.

In geometrical visualization of $\mathcal{K}$ one can exhibit how a combinatorial vector field $\mathcal{V}$ is defined on $\mathcal{K}$ by means of arrows and dots. For σ and τ in $\mathcal{K}$, there is an arrow from σ to τ if $\mathcal{V}(\sigma) = \tau$ and $\sigma \neq \tau$, and there is a dot at the centroid of σ if $\mathcal{V}(\sigma) = \sigma$.

A sequence of cubes $\sigma_1, \tau_1, \ldots, \sigma_n, \tau_n$ in $\mathcal{K}$ is called a $\mathcal{V}$-*path* of index p if

(i) σ_i and τ_i has dimension p and $p+1$ respectively for $i = 1, \ldots, n$,

(ii) $\tau_i = V(\sigma_i)$,

(iii) $\sigma_i \neq \sigma_{i+1}$ and σ_{i+1} is a facet of τ_i.

A *loop* in $\mathcal{K}$ is a closed $\mathcal{V}$-path. That is a $\mathcal{V}$-path $\sigma_1, \tau_1, \ldots, \sigma_n, \tau_n$ with $\sigma_1 = \sigma_n$ and $\tau_1 = \tau_n$.

For a non-trivial loop in $\mathcal{K}$ which contains simplices of dimension p and $p+1$, the index of the loop is taken to be p and its dimension is taken to be $p+1$.

The final goal is to define Conley index for a finite simplicial complex $\mathcal{K}$ equipped with a combinatorial vector field $\mathcal{V}$. But to achieve that, a possible approach is to construct an upper semi-continuous map with non-empty and contractible values F that is defined on geometric realization of $\mathcal{K}$ denoted by $|\mathcal{K}|$. The multivalued map F is essential in defining index pair in this approach. This work has been elaborately described in [13], [2], [4]. The auxiliary map which makes connection between combinatorial vector field $\mathcal{V}$ and multi-valued map F is the flow $\Pi_{\mathcal{V}}$ associated with $\mathcal{V}$, that is defined as:

$$\Pi_{\mathcal{V}}(\sigma) = \begin{cases} Cl(\sigma) & \text{if } \sigma \in \text{Fix } \mathcal{V}, \\ Bd(\sigma) \backslash \{\mathcal{V}^{-1}(\sigma)\} & \text{if } \sigma \in \text{Im } \mathcal{V} \backslash \text{Fix } \mathcal{V}, \\ \{\mathcal{V}(\sigma)\} & \text{if } \sigma \in \text{Dom } \mathcal{V} \backslash \text{Fix } \mathcal{V}. \end{cases} \tag{2.10}$$

The set $\mathrm{Cl}(\sigma)$ denotes the combinatorial closure of σ which is the set including σ and all its faces. The set $\mathrm{Bd}(\sigma)$ is the combinatorial boundary of the cube σ which is $\mathrm{Cl}(\sigma)\backslash\{\sigma\}$.

To get around the limitation with dimension of simplices in orbits directly generated by a combinatorial vector field $\mathcal{V}$, one can take orbits of the flow associated with $\mathcal{V}$ in Equation 2.10.

For the convenience in proofs and discussions, the following notions will be useful:

$$\sigma^+ := \begin{cases} \mathcal{V}(\sigma) & \text{if } \sigma \in \operatorname{dom}\mathcal{V} \\ \sigma & \text{otherwise} \end{cases} \quad \text{and} \quad \sigma^- := \begin{cases} \sigma & \text{if } \sigma \in \operatorname{dom}\mathcal{V} \\ \mathcal{V}^{-1}(\sigma) & \text{otherwise} \end{cases}. \tag{2.11}$$

A *solution* or an *orbit* of the multi-valued map $\Pi_{\mathcal{V}}$ is a partial function $\varrho : \mathbb{Z} \nrightarrow \mathcal{K}$ such that its domain is an interval in $\mathbb{Z}$, and

$$\forall i \in \mathrm{Dom}(\varrho) \text{ with } i+1 \in \mathrm{Dom}(\varrho),\ \ \varrho(i+1) \in \Pi_{\mathcal{V}}(\varrho(i)). \tag{2.12}$$

A solution $\varrho : \mathbb{Z} \to \mathcal{K}$ is called a *full solution* if $\mathrm{Dom}(\varrho) = \mathbb{Z}$.

This generalizes the notion of $\mathcal{V}$-path given by Forman in [9]. The full solutions under the flow $\Pi_{\mathcal{V}}$ admits asymptotic behavior in combinatorial setting, moreover, full solutions will be possible and defined, even when simplicial complex is finite.

Having the notion of full solution and its possibility in a finite simplicial complex now allows us to talk about invariant sets under the flow $\Pi_{\mathcal{V}}$.

Definition 2.4. *Let $\mathcal{K}$ be a finite simplicial complex and $\mathcal{V}$ be a combinatorial vector field on $\mathcal{K}$. Then, the set $\mathcal{S} \subset \mathcal{K}$ is called invariant under associated flow $\Pi_{\mathcal{V}}$, if for each $\sigma \in \mathcal{S}$, there exists a full solution $\varrho : \mathbb{Z} \to \mathcal{K}$ within $\mathcal{S}$ which passes through σ, i.e., $\varrho(k) = \sigma$ for some $k \in \mathbb{Z}$ and $\varrho(\mathbb{Z}) \subset \mathcal{S}$.*

A particular type of invariant sets, are called isolated invariant sets. That is an essential notion in Conley index theory. In time continuous dynamics, an isolated invariant set S has the property that an orbit originating from its interior will not make internal tangency to the boundary of the set S. Analogy of this notion in combinatorial dynamics, is introduced in [13] as follows.

Definition 2.5. *Let $\mathcal{K}$ be a finite simplicial complex and $\mathcal{V}$ be a combinatorial vector field on $\mathcal{K}$. Furthermore, let $S \subset \mathcal{K}$ denote an invariant set under the flow $\Pi_{\mathcal{V}}$. Exit set of $\mathcal{S}$ is defined by*

$$Ex(\mathcal{S}) = Cl(\mathcal{S}) \backslash \mathcal{S}. \tag{2.13}$$

Then, the invariant set $\mathcal{S}$ is called an isolated invariant set, if the following conditions hold:

(i) The exit set $Ex(\mathcal{S})$ is closed in simplicial complex $\mathcal{K}$,

(ii) There exists no solution $\varrho : \{-1, 0, 1\} \to$ on $\Pi_{\mathcal{V}}$ such that both $\varrho(-1) \in \mathcal{S}$, $\varrho(1) \in \mathcal{S}$, and $\varrho(0) \in Ex(\mathcal{S})$.

A subset $\mathcal{A}$ of a simplicial complex $\mathcal{K}$ is called *closed* if for every $\sigma \in \mathcal{A}$, all faces of σ also belong to $\mathcal{A}$.

One can find examples in [13] which show how failing to satisfy one of the two above conditions results in internal tangency of the associated background time continuous dynamic. That means the combinatorial invariant set does not correspond to an isolated invariant set in the underlying time-continuous dynamical system.

Practically, the second condition in Definition 2.5 is usually hard or time consuming to verify. It can be replaced by a more practical and visible condition. Assume $\mathcal{V}$ is a combinatorial vector field on a finite simplicial complex $\mathcal{K}$. Moreover assume $\mathcal{S} \subset \mathcal{K}$ is an isolated invariant set under the flow $\Pi_{\mathcal{V}}$. Then for every $\sigma \in \mathcal{K}$, we have $\sigma^- \in \mathcal{S}$ iff $\sigma^+ \in \mathcal{S}$. That means σ^- and σ^+ either both are in S or both are outside of S. This results is an alternative definition for notion of isolation in combinatorial setting.

Proposition 2.6. *An invariant set $\mathcal{S}$ is an isolated invariant set if it satisfies the following two conditions:*

(i) The exit set $Ex(\mathcal{S})$ is closed in simplicial complex $\mathcal{K}$,

(ii) for every $\sigma \in \mathcal{K}$, $\sigma^- \in \mathcal{S}$ iff $\sigma^+ \in \mathcal{S}$.

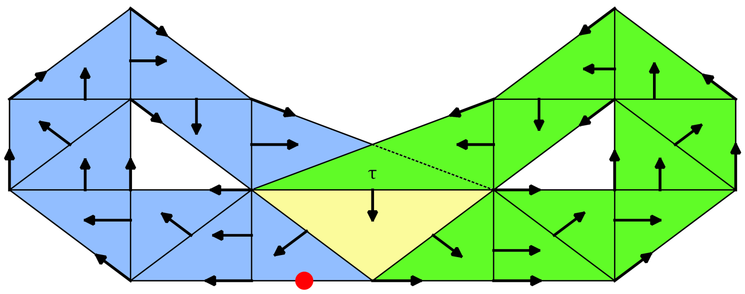

Figure 2.1: A model for combinatorial Lorenz system in simplicial setting.

2.3 Chaos in Combinatorial Lorenz Model

Earlier in this chapter we referenced [17] and [19] for proof of the chaos in Lorenz system for an open set of parameters. The task is done by means of Poincaré section and demonstration for existence of a conjugacy between the invariant set of the Poincaré section and the symbolic space in two symbols.

A model for combinatorial Lorenz system in simplicial setting was suggested in [13] which is shown in Figure 2.1. It will be interesting to transfer a similar idea for proof of chaos in continuous time to combinatorial Lorenz system by means of conjugacy, to prove the chaos in this combinatorial model.

Before the main theorem for proof of chaos in combinatorial Lorenz system, we look further into the structure of this model. That could give us intuition about what to take as section, Poincaré section and the orbit which produces the Poincaré map. Let's consider two particular non-homotopic loops in combinatorial Lorenz model shown in Figure 2.2. These two loops can be thought as periodic orbits and they have the τ in common shown in Figure 2.1. On left or right loop once we arrive at the edge τ one can switch from one loop to the other. Let L_1 and L_2 denote the left and right loop shown in Figure 2.2 respectively. Then L_1 and L_2 can follow each other as a part of a full solution in any arbitrary order. It is just sufficient to decide between L_1 and L_2 once we arrive at τ. Looking only at indices of loops L_1 and L_2 as a bi-infinite sequence, we get an element of Σ_2. That means this combinatorial model for the Lorenz system contains a dynamic similar to Σ_2. Therefore, dynamical behavior of combinatorial Lorenz system is at least as complicated as the dynamics of symbol space in two symbols. Therefore, this

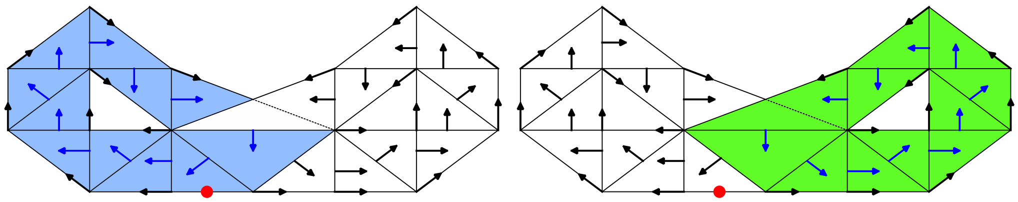

Figure 2.2: Two non-homotopic periodic orbits of index 1 in combinatorial Lorenz model shown in blue and green color.

combinatorial dynamical system is chaotic.

Let $\mathcal{L}$ denote the simplicial complex with the geometric realization shown in Figure 2.1. The arrows and the red dot in Figure 2.1 defines a combinatorial vector field on $\mathcal{L}$. The edge marked by a red dot is the only critical simplex. The pair $(\mathcal{L}, \mathcal{V})$ of the simplicial complex $\mathcal{L}$ with the vector field $\mathcal{V}$ defined on it, is a combinatorial model for Lorenz system.

Next, let $A \subset \mathbb{R}^2$ be the support of the left periodic orbits of index 0 in Figure 2.1 consisting of three edges and three vertices, and similarly the $B \subset \mathbb{R}^2$ be the support of the right periodic orbits of index 0 consisting of three edges and three vertices. Assume $\varrho : \mathbb{Z} \to \mathcal{L}$ is a full solution of the flow $\Pi_{\mathcal{V}}$. For simplicity we denote it by $\{\sigma_n\}_{n\in\mathbb{Z}}$. Consider a subsequence $\{\sigma_n\}_{n_1<n<n_2}$ of $\{\sigma_n\}_{n\in\mathbb{Z}}$ such that $|\sigma_{n_1}|$, the geometric realization of σ_{n_1} and $|\sigma_{n_2}|$, the geometric realization of σ_{n_2} have non-empty intersection. Now we consider the geometrical realization of the simplicial complex $\bigcup\limits_{i=n_1}^{n_2} \{\sigma_i\}$ which can be homotopy equivalent to A or B or none. Every full solution $\varrho : \mathbb{Z} \to \mathcal{L}$, simply denoted by $\{\varrho(n) = \sigma_n\}_{n\in\mathbb{Z}}$, can be partitioned into sub-solutions $\varrho_i : I_i \to \mathcal{L}$, where

- $I_i = [n_i, n_{i+1}]$ with $n_i < n_{i+1}$ is an interval in $\mathbb{Z}$,
- $|\sigma_{n_i}| \cap |\sigma_{n_{i+1}}| \neq \emptyset$,
- $\bigcup\limits_{i=n_i}^{n_{i+1}} \{\sigma_i\}$ is not contractible,
- n_{i+1} is minimal with respect to preceding conditions.

Intervals $\{I_i\}_i$ splits $\mathbb{Z}$ into non-empty, single-point-overlapping intervals. The geometric realization of each of this sub-solutions is either homotopic to A or B.

Let $\mathcal{P}(\mathcal{L})$ denote the set of all full solutions of $(\mathcal{L}, \mathcal{V})$ under $\Pi_{\mathcal{V}}$ which do not pass through the critical edge. Then, every solution $\varrho \in \mathcal{P}(\mathcal{L})$ can be partitioned into sub-solutions whose geometric realization of each is either homotopy equivalent to A or B. For such sequence of sub-solution, one can associate a bi-infinite sequence consisting of 1 and 2, where the i'th term of this sequence is 1 or 2 if the sub-solution on $[n_i, n_{i+1}]$ is homotopy equivalent to A or B respectively, which is indeed an element of Σ_2. We denote this sequence obtained from ϱ by seq(ϱ).

Now, we define a relation R on $\mathcal{P}(\mathcal{L})$ given by

$$\forall \varrho_1, \varrho_2 \in \mathcal{P}(\mathcal{L}) \quad \varrho_1 R \varrho_2 \text{ iff } \mathrm{seq}(\varrho_1) = \mathrm{seq}(\varrho_2). \tag{2.14}$$

Then R is an equivalence relation on $\mathcal{P}(\mathcal{L})$.

Similar argument can be made for the set of infinite forward time solutions on $(\mathcal{L}, \mathcal{V})$ under $\Pi_{\mathcal{V}}$ which do not pass through the critical edge. We denote the set of such solutions by $\mathcal{P}^+(\mathcal{L})$. The relation given in Equation 2.14 is an equivalence relation on $\mathcal{P}^+(\mathcal{L})$. By a misuse of notation, $\mathcal{P}^+(\mathcal{L})/R$, the set of all equivalent classes of $\mathcal{P}^+(\mathcal{L})$ with respect to R, is still denoted by $\mathcal{P}^+(\mathcal{L})$ and we call it *space of forward loop itineraries* on $(\mathcal{L}, \mathcal{V})$.

Theorem 2.7. *Let $(\mathcal{L}, \mathcal{V})$ denote the combinatorial model for Lorenz system shown in Figure 2.1 and $\mathcal{P}^+(\mathcal{L})$ be the space of forward loop itineraries on $(\mathcal{L}, \mathcal{V})$. Moreover, assume the shift map $s : \Sigma_2^+ \to \Sigma_2^+$ is given by,*

$$s\big((a_0, a_1, a_2, \ldots)\big) = (a_1, a_2, \ldots). \tag{2.15}$$

Then, there are maps $f : \mathcal{P}^+(\mathcal{L}) \to \mathcal{P}^+(\mathcal{L})$ and an invertible map $h : \mathcal{P}^+(\mathcal{L}) \to \Sigma_2^+$ such that $h \circ f = s \circ h$.

Proof. Let τ be the edge shown in Figure 2.1 and consider $|\tau|$ as interval $[0, 1]$. For any $r \in [0, 1]$ there is a unique representation in binary base of the form $r = (0.r_0 r_1 r_2 \ldots)_2$, where $r_i \in \{0, 1\}$ for $i \in \mathbb{N} \cup \{0\}$. For the uniqueness of this representation, it is required that the decimal representation of r in base 2 is infinite. In other words if $r = (0.r_0 r_1 \ldots r_k)_2$ with $r_k = 1$, we consider the representation of r in base 2 as $(0.r_0 r_1 \ldots r_{k-1} 011 \ldots)_2$. Now, for every $x_0 \in$

$|\tau|$ we assume $x_0 = (0.a_0a_1a_2\ldots)_2$ is its infinite binary representation. We can uniquely associate an orbit $\gamma_{x_0} : \mathbb{Z}^{\geq 0} \to \mathcal{L}$ where the orbit initiates at x_0, turns around left loop if $a_0 = 0$ and turns around right loop if $a_0 = 1$. Then for next turn we look at a_1, if it is 0 or 1 we turn around left loop or right loop respectively and keep doing this procedure for the rest of a_i's. We define the map f to be $f((0.a_0a_1a_2\ldots)_2) = (0.a_1a_2\ldots)_2$. Furthermore, define h to be $h((0.a_0a_1a_2\ldots)_2) = (a_0, a_1, a_2, \ldots)$. Then the map h is invertible, moreover for every $x_0 \in |\tau|$ with binary representation $x_0 = (0.a_0a_1a_2\ldots)_2$ we have,

$$(h \circ f)(x_0) = h \circ (f((0.a_0a_1a_2\ldots)_2)) = h((0.a_1a_2\ldots)_2) = (a_1, a_2, \ldots), \quad (2.16)$$

on the other hand

$$(s \circ h)(x_0) = s \circ (h((0.a_0a_1a_2\ldots)_2)) = s(a_0, a_1, a_2\ldots) = (a_1, a_2, \ldots) \quad (2.17)$$

Therefore, $h \circ f = s \circ h$. $\square$

The proof of chaos for combinatorial model of Lorenz system using the notion of Poincaré section and semi-conjugacy does not seem to apply here. One of the reasons could be that, one requires a Poincaré section and an orbit on $\mathcal{L}$ with the associated Poincaré map as g, and a surjective map $\varrho : \text{Inv}(g, N) \to \Sigma_2$. The set Σ_2 is uncountable while the set $\text{Inv}(g, N)$ is finite from the combinatorial point of view. Therefore, a surjection $\varrho : \text{Inv}(g, N) \to \Sigma_2$ does not exist.

2.4 Morse Decomposition of Combinatorial Lorenz System Models

Assume $\mathcal{K}$ be a finite simplicial complex and assume $\mathcal{V}$ be a combinatorial vector field on $\mathcal{K}$ and $\Pi_{\mathcal{V}}$ denote the flow associated with $\mathcal{V}$. Assume $\varrho : \mathbb{Z} \to \mathcal{K}$ is a full solution under $\Pi_{\mathcal{V}}$. Analogous to continuous time dynamical systems the α and ω-limit sets are defined as following.

$$\alpha(\varrho) = \bigcap_{k \in \mathbb{N}} \varrho((-\infty, -k]), \qquad \omega(\varrho) = \bigcap_{k \in \mathbb{N}} \varrho([k, \infty)). \quad (2.18)$$

The α and ω-limit sets are always non-empty invariant sets since $\mathcal{K}$ is finite.

Morse Decomposition of time-continuous dynamical system is given in Definition 1.9. Combinatorial Morse decomposition is defined as follows.

Definition 2.8. *Let $\mathcal{V}$ be a combinatorial vector field on a finite simplicial complex $\mathcal{K}$, and let $\Pi_{\mathcal{V}}$ denote the flow associated with $\mathcal{V}$. A Morse decomposition of $\mathcal{K}$ is a family $\{\mathcal{M}_\lambda \mid \lambda \in \Lambda\}$ indexed by a poset Λ which satisfies the following conditions:*

1. *The sets $\mathcal{M}_\lambda$ are mutually disjoint isolated invariant subsets of $\mathcal{K}$ under the flow $\Pi_{\mathcal{V}}$.*
2. *For every full solution φ of flow $\Pi_{\mathcal{V}}$, there are $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1 > \lambda_2$ such that $\alpha(\varphi) \subset \mathcal{M}_{\lambda_1}$ and $\omega(\varphi) \subset \mathcal{M}_{\lambda_2}$.*
3. *If for a full solution φ of flow $\Pi_{\mathcal{V}}$ and $\lambda \in \Lambda$, we have $\alpha(\varphi) \cup \omega(\varphi) \subset \mathcal{M}_\lambda$, then $\varphi(\mathbb{Z}) \subset \mathcal{M}_\lambda$.*

For every Morse decomposition $\{\mathcal{M}_\lambda \mid \lambda \in \Lambda\}$ of $\mathcal{K}$, one can associate a digraph. The nodes of this digraph are the sets $\mathcal{M}_\lambda$ and there is directed edge from $\mathcal{M}_{\lambda_1}$ to $\mathcal{M}_{\lambda_2}$ iff there is a full solution φ with $\alpha(\varphi) \in \mathcal{M}_{\lambda_1}$ and $\omega(\varphi) \in \mathcal{M}_{\lambda_2}$.

A digraph is short term for directed graph, and it is a diagram composed of points called vertices (nodes) and arrows called edges (arcs) going from a vertex to a vertex. Formally, a digraph is an ordered pair of sets $G = (V(G), E(G))$, where $V(G)$ is a set of vertices and $E(G)$ is a set of ordered pairs(directed edges)of vertices of $V(G)$.

In this section we try to find a non-trivial Morse decomposition for the combinatorial Lorenz system shown in Figure 2.1. Ideally, we expect the set of left loop, right loop shown Figure 2.2 and the critical edge form a Morse decomposition for this system, namely L, R, C. According to the definition, a decomposition set should be an isolated invariant set. That means, it is invariant, its exit set is closed, and tail of an arrow is in the set iff its head is in the set. Some possibilities for L and R are shown in Figure 2.3. But in each of the those three cases, either the invariant condition fails, or the isolation condition fails as exit set is not closed in some choices of L or R as a decomposition set. That suggests that maybe a modification in the model can help to get a finer decomposition, ideally a Morse decomposition, for the system containing the left and right loop as well as the critical edge. A new model is suggested for combinatorial Lorenz in simplicial complex as well as similar model for Lorenz system in cubical setting which has the left and right loop of index 1 as disjoint isolated invariant sets. We may take $\{L, R, C\}$ as left and right loops of index 0 and critical edge, respectively but it

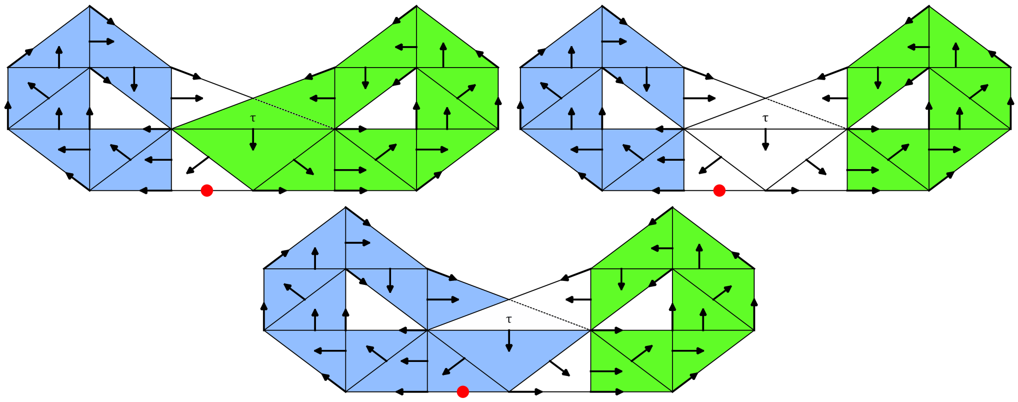

Figure 2.3: Attempts to make a non-trivial decomposition for Lorenz combinatorial model in simplicial complex setting. The green and blue colors are meant to represent a combinatorial invariant set.

is not a Morse decomposition for the Lorenz system shown in Figure 2.1. We discuss this in more details in next section.

2.4.1 Morse Decomposition of Combinatorial Lorenz System in Simplicial Setting

For the combinatorial Lorenz system in Figure 2.1, we may consider the left and right index 0 loops and the critical edge as a family of mutually disjoint isolated invariant sets, but that fails to satisfied the third condition in Definition 2.8. In fact, some orbits originating from critical edge can turn around the loops arbitrarily without terminating at any of the three decomposition sets. This is in contrast with the second condition in Definition 1.9. To prevent such orbits which do not terminate at any decomposition set, we may consider left and right loops of index 1. But still is not possible to have a Morse decomposition which has the left and right loops of index 1, and the critical edge as a Morse decomposition. The left and right loop of index 1 are invariant but not mutually disjoint. In Figure 2.3 we can see some possibilities for obtaining disjoint isolated invariant sets from left and right loops of index 1. In Figure 2.3, one can see reducing the loops to a smaller set to make them disjoint, encounters with some problems. Either the set is not invariant or it it is invariant but not isolated.

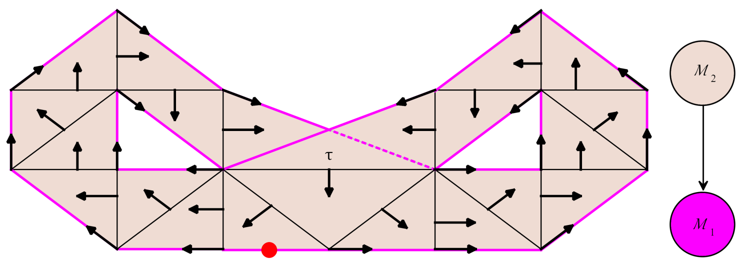

Figure 2.4: Coarsest non-trivial Morse decomposition for the original combinatorial Lorenz system in simplicial complex setting and its associated digraph.

The coarsest non-trivial Morse decomposition for the Lorenz system $\mathcal{L}$ in Figure 2.1 is the decomposition $\{\mathcal{M}_1, \mathcal{M}_2\}$ where $\mathcal{M}_1$ is the critical edge union the left and right loops of index 0 and the index 0 paths which connect the critical edge to index 0 loops, and $\mathcal{M}_2$ is the rest of the complex. We define the partial order $<$ on the index set $\{1, 2\}$ to be $1 < 2$. It is easy to verify that $\mathcal{M}_1$ and $\mathcal{M}_2$ are mutually disjoint isolated invariant sets. Also the other two conditions in Definition 2.8 are satisfied. In Figure 2.4 the Morse decomposition and Morse graph are shown.

A finer Morse decomposition for Figure 2.1 is obtained by splitting the Morse set $\mathcal{M}_1$ in previous decomposition into three mutually disjoint isolated invariant sets. The new components are shown in purple, green and blue color in Figure 2.5 and they are clearly isolated invariant sets. In this case the Morse decomposition is $\{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4\}$ with the partial order

$$1 < 3,\ 2 < 3,\ 3 < 4,\ 1 < 4,\ 2 < 4$$

on the index set $\{1, 2, 3, 4\}$ and the Morse sets, $\mathcal{M}_3$ to be the critical edge, $\mathcal{M}_1$ and $\mathcal{M}_2$ the left and right loops of index 0, and $\mathcal{M}_4$ the union of the left and right loops of index 1.

Another combinatorial dynamical system which we study in this section is shown in Figure 2.6 and it is analogous to the dynamical system in Figure 2.1. Even though it exhibits chaotic behavior similar to Figure 2.1, their Morse decompositions are different and the new model possesses extra critical simplices.

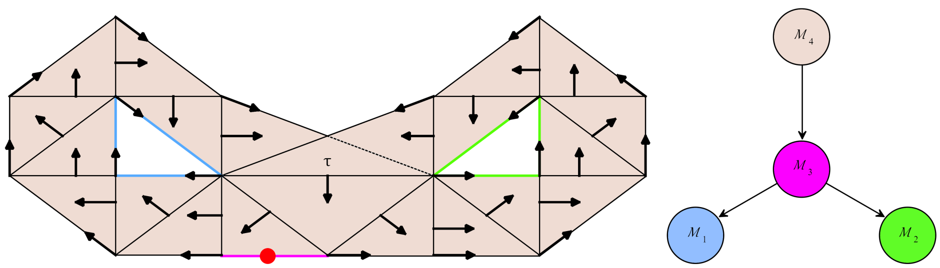

Figure 2.5: A finer Morse decomposition for the original combinatorial Lorenz system in simplicial complex stetting and its associated digraph.

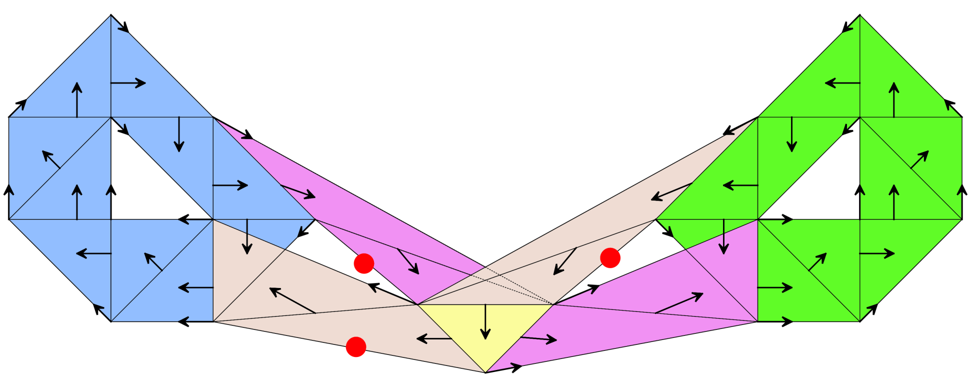

Figure 2.6: Modified combinatorial Lorenz system in simplicial complex model. The light brown region connecting the green region to the yellow region and the purple region connecting blue region to the yellow region, only intersect at the top edge of the yellow region.

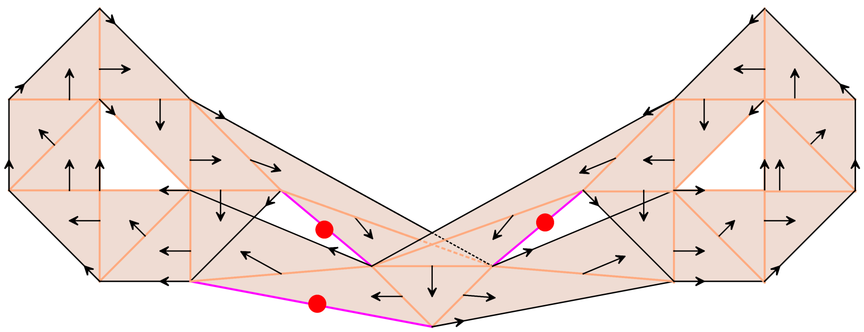

Figure 2.7: An attempt for making a Morse decomposition for the modified combinatorial Lorenz system in simplicial complex model. The union of index 1 loops shown in brown color, and the three edges in purple color each is an isolated invariant sets.

In the rest of this section we study the Morse decomposition of the modified combinatorial Lorenz model given in Figure 2.6. The initial motivation to construct this model is that, the left and right loops of index 1 can be taken as disjoint isolated invariant sets while this is not possible in the original combinatorial Lorenz model, as shown in Figure 2.3. We try to decompose the system given in Figure 2.6 into mutually disjoint isolated invariant sets and verify which of these decompositions is a Morse decomposition.

The two loops of index 0 and three critical points are mutually disjoint isolated invariant sets, but this is not a Morse decomposition, since some orbits will not terminate at any of these decomposition sets. Now, if we take one of isolated invariant sets $\mathcal{M}_1$ be the index 1 path in brown color shown in Figure 2.7, also the three critical edges as the other isolated invariant sets, it results in a trivial Morse decomposition. Because of existence of a full solution through $\mathcal{M}_1$ from a critical edge to itself, the third condition in Definition 2.8 imposes that all three critical edges must belong to $\mathcal{M}_1$. The rest of simplices in the system must be also included in the Morse set as well, since for any remained simplex σ, there is a full solution from $\mathcal{M}_1$ back to $\mathcal{M}_1$ passing through σ.

Another decomposition of the system in Figure 2.6 is the family of sets, union of the left loops of index 0 and 1, union of the right loops of index 0 and 1, and the three critical edges. This is shown in Figure 2.8 as the top image, but it is not

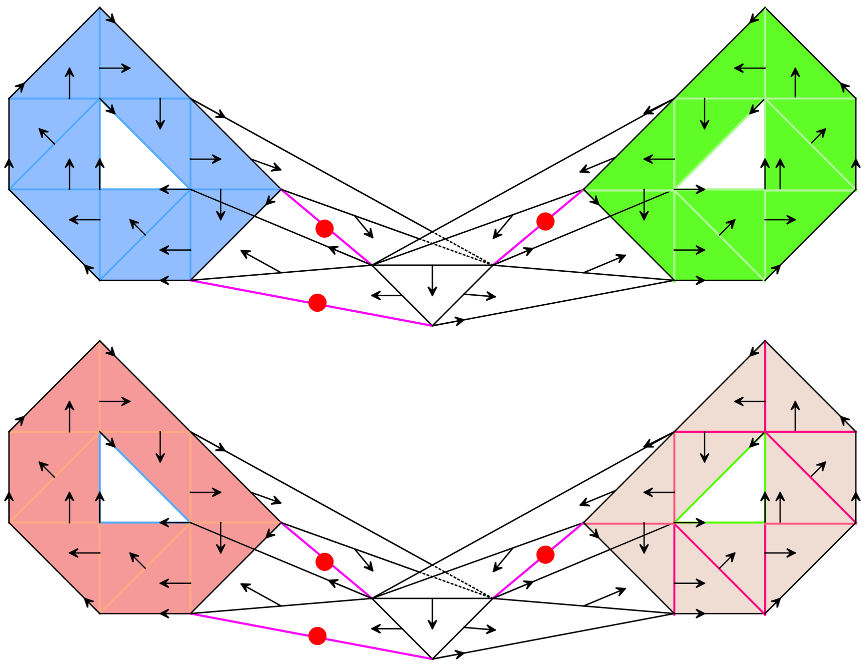

Figure 2.8: Two decompositions of the alternative combinatorial Lorenz system into mutually disjointed isolated invariant which none is a Morse decomposition.

a Morse decomposition as the third condition in Definition 2.8 fails. As a finer Morse decomposition, in preceding decomposition, we can take the index 0 and 1 loops as distinct decomposition sets which is shown in Figure 2.8 as the bottom image, and this decomposition is not a Morse decomposition either due to the same reason as in the previous decomposition.

Finally, if we consider the preceding decomposition and join the left and right loops of index 1 and the index 1 paths which connect those two loops, all as one decomposition set, and the rest of the decomposition sets stay the same, then we obtain a Morse decomposition. The Morse sets for this decomposition as well as the associated Morse graph to it are shown in different colors in Figure 2.9.

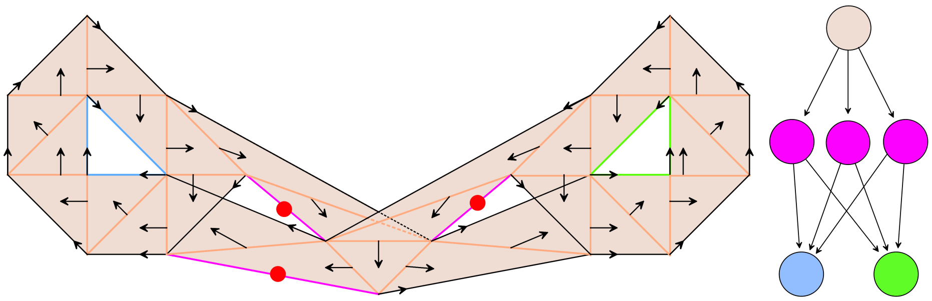

Figure 2.9: A Morse decomposition for the modified combinatorial Lorenz system in simplicial complex setting. The light brown, purple, green, blue colors demonstrate the Morse sets.

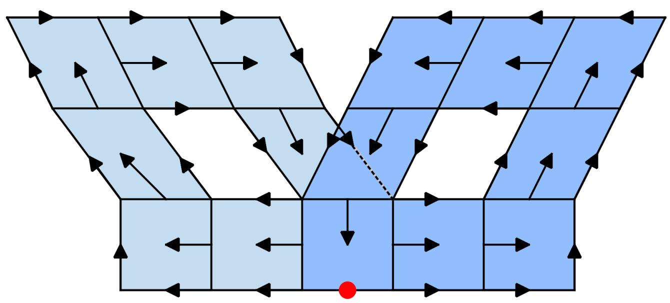

Figure 2.10: A model for combinatorial Lorenz system in cubical setting

2.4.2 Morse Decomposition of Combinatorial Lorenz System in Cubical Setting

In the next chapter we will discuss dynamical systems in cubical setting. An example of a dynamical system in cubical setting is shown in Figure 2.10 which can be considered as a combinatorial model for Lorenz System.

An alternative combinatorial model for Lorenz system in cubical setting is shown in Figure 2.11. The two models given in Figure 2.10 and Figure 2.11 demonstrate chaotic behavior but they possess different Morse graphs. The later model allows the left and right loops of index 1 form disjoint isolated invariant sets, while it is not possible in former model.

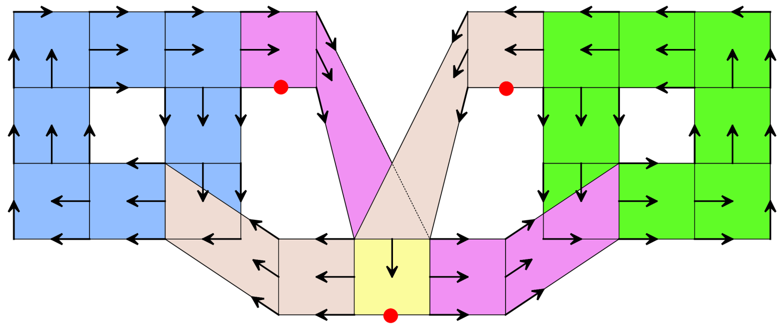

Figure 2.11: An alternative model for combinatorial Lorenz system in cubical setting.

Analogous arguments made in previous section for Lorenz model in simplicial complex setting, in particular the chaos and the Morse decomposition, could be done for the above two models of Lorenz system in cubical setting. Since the results will be similar to those in simplicial setting, we skip the repetition.

2.5 Minimal Morse Decomposition of a Combinatorial Dynamical System

In [7], it is shown how to construct a multi-vector field associated with a dynamical system. The method is based on taking sample points from the dynamical system with a certain sampling frequency. The construction of multi-vector field is dependent on threshold parameter which can help to eliminate the noise in the sampling data. Moreover, for a combinatorial dynamical system a digraph is associated with it which can translate some combinatorial concepts into digraph concepts.

Let $\mathcal{K}$ be a simplicial complex and let $\Gamma : \mathcal{K} \multimap \mathcal{K}$ denote a combinatorial multi-valued map. The digraph associated with Γ is denoted by G_Γ and is defined as follows. The vertices of G_Γ are simplices in $\mathcal{K}$ and there is a directed edge from σ to τ if and only if $\tau \in \Gamma(\sigma)$. Some combinatorial dynamical system concepts can be translated into concepts in digraphs. For instance, a partial solution in $\mathcal{K}$ under Γ is a walk in digraph G_Γ. A solution under Γ in $\mathcal{A} \subset \mathcal{K}$ is a walk in G_Γ with all vertices in $\mathcal{A}$. Moreover, a set $\mathcal{A} \subset \mathcal{K}$ is invariant if every vertex of G_Γ which is in $\mathcal{A}$ is incident to a bi-infinite walk in G_Γ through vertices in $\mathcal{A}$. Concept of isolation for an invariant set given in [7] can not be translated into a digraph concept only. However, the concept of minimal Morse decomposition can be expressed in language of digraphs. A minimal Morse decomposition for a combinatorial dynamical system is a Morse decomposition which none of its Morse sets has a non-trivial Morse decomposition. A *strongly connected component* of a digraph G is a subgraph $\tilde{G}$ of G which for every two vertices u and v in $\tilde{G}$ there is a directed path from u to v. The following theorem from [7] proves the existence and uniqueness of the minimal Morse decomposition of a combinatorial dynamical systems by means of its associated digraph.

Theorem 2.9. *Assume $\mathcal{K}$ is a simplicial complex and let $\Gamma : \mathcal{K} \multimap \mathcal{K}$ be a combinatorial multi-valued map. Moreover, let G_Γ be the digraph associated with Γ. The family of all strongly connected components of G_Γ is the unique minimal Morse decomposition of the maximal invariant set of $\mathcal{K}$ under Γ.*

There are several algorithms for discovering strongly connected components of a directed graph. Among the most well known algorithms, we may name Kosaraju's algorithm independently suggested by Kosaraju and Sharir in 1979, and Tarjan's algorithm suggested by Robert Endre Tarjan. Here we recall both algorithms (ref.[6]).

Tarjan's algorithm uses a recursive depth first search (DFS) to form a search tree of explored vertices. The roots of the subtrees of the search tree form roots of strongly connected components (SCC).

Kosaraju's algorithm performs two passes of the graph. It initially performs a DFS, placing each vertex onto a stack after it has been fully explored. After all vertices have been placed onto the stack, a vertex is popped from the stack and a DFS or breadth first search (BFS) search is performed on the transpose of the graph. All vertices that can be reached by this vertex (that have not already been explored a second time) form an SCC. The runtime for both algorithms is in $O(n + m)$ time, where n is the number of vertices and m is the number of edges in a graph. However, since Tarjan's algorithm only requires a single search as apposed to Kosaraju's two, it is often faster in practice. We have chosen to apply Kosaraju's algorithm here as it is more convenient to work with manually.

Here we apply Kosaraju's algorithm on two combinatorial models of Lorenz system shown in Figure 2.1 and Figure 2.6 to obtain the SSC of the model. Then, Theorem 2.9 guarantees that the Morse decomposition we obtained earlier for the two combinatorial models of Lorenz system are really their minimal Morse decomposition.

Assume $\mathcal{V}$ is a combinatorial vector field defined on a finite simplicial complex $\mathcal{X}$ and let $\Pi_{\mathcal{V}}$ be the associated flow to $\mathcal{V}$. The digraph $G_{\mathcal{X},\mathcal{V}} = (V(G), E(G))$ associated with the system $(\mathcal{X}, \mathcal{V})$, has a vertex associated with each simplex in $\mathcal{X}$, and there is a directed edge from the vertex associated with σ to the vertex associated with τ iff

$$\tau \in \Pi_{\mathcal{V}}(\sigma). \tag{2.19}$$

First step before applying Kosaraju's algorithm is to obtain the digraph associated with each of the two combinatorial models of Lorenz system. We start with the system given in Figure 2.1.

The simplex associated in the model Figure 2.1 has 19 simplices of dimension zero, 41 simplices of dimension one, and 21 simplices of dimension two. Therefore the digraph associated with this system has 81 vertices and the directed edges between two vertices follow the rule in Equation 2.19. In drawing directed edges in the associated digraph without aid of a computer program code one can easily miss an edge or make an error. Here we try to reduce the digraph associated with this system first and the we apply the Kosaraju's algorithm to detect the SCC.

The sub-digraph associated with the left loop of index 1 is a strongly connected component, so is that of the right loop of index 1 and they intersect so they belong to one strongly connected component. Having that component discovered, we now remove the union of left and right loops of index 1 which results in a less complicated system shown in Figure 2.12.

The digraph associated with the model in Figure 2.12 is given in Figure 2.13 and we call it G_1.

Now we apply DFS to G_1 with initial vertex to be σ_1 which result is the starting and finishing time pair for each vertex of G_1 shown in Figure 2.14. The Kosaraju's algorithm works independent of choice of the initial vertex.

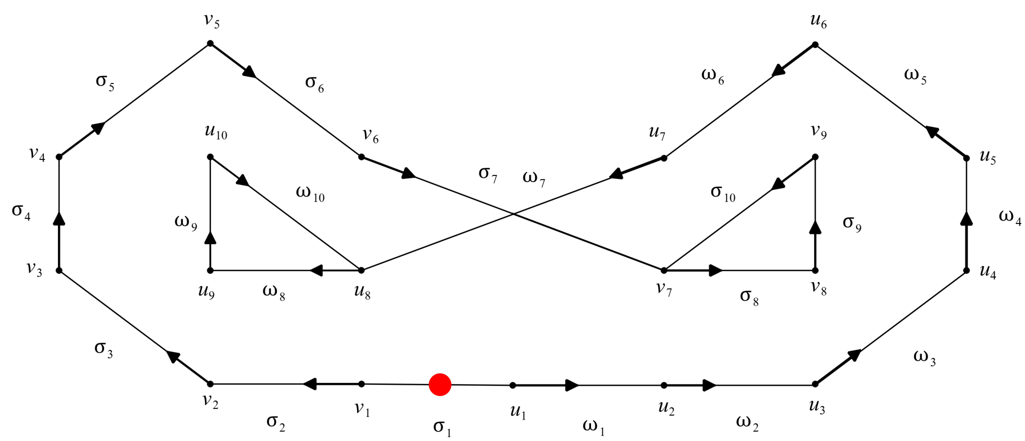

Figure 2.12: Left and right loops of index 1 removed from Lorenz combinatorial model.

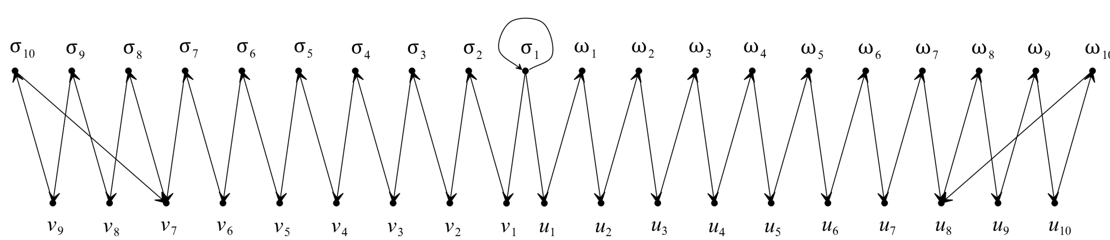

Figure 2.13: The digraph associated with the system shown in Figure 2.12.

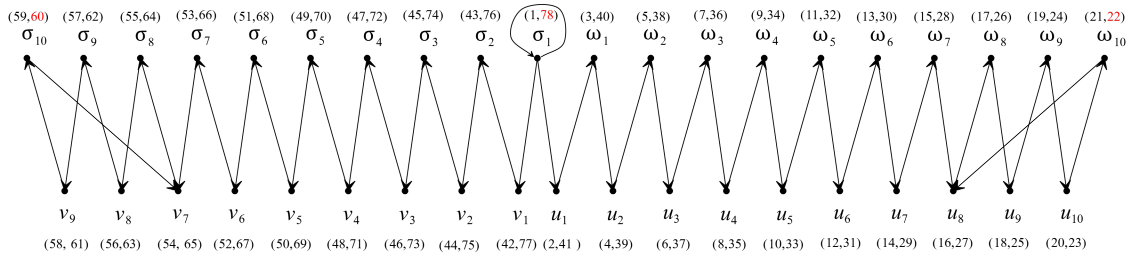

Figure 2.14: Starting and finishing time associated with the system shown in Figure 2.13.

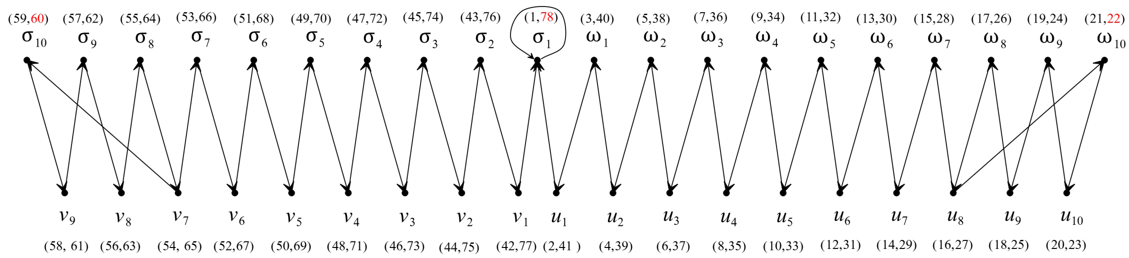

Figure 2.15: Second stage of Kosaraju's algorithm: DFS applied to the transpose of the digraph in Figure 2.14.

The next step is to construct the transpose of G_1, denoted by ${G_1}^t$ which has the same vertices as G_1 and vertex u is connected to v in ${G_1}^t$ if v is connected to u in G_1. The graph ${G_1}^t$ is shown in Figure 2.15. Next, we sort the vertices of G_1 in a decreasing order with respect to the finishing time and putting them in stack. We take vertices from the stack one by one, starting with the highest finishing time and then we apply DFS to that vertex to get the first SCC. Then, we remove the vertices of the found SCC from the stack. The procedure should be repeated until the stack is empty. The outcome of the DFS on ${G_1}^t$ is:

$$\begin{aligned}&\{\sigma_1, \sigma_1\}, \{v_1\}, \{\sigma_2\}, \cdots, \{v_6\}, \{\sigma_7\}, \{v_7, \sigma_{10}, v_9, \sigma_9, v_8, \sigma_8\},\\ &\{u_1\}, \{\omega_1\}, \cdots, \{u_7\}, \{\omega_7\}, \{u_8, \omega_{10}, u_{10}, \omega_9, u_9, \omega_8\}\end{aligned} \tag{2.20}$$

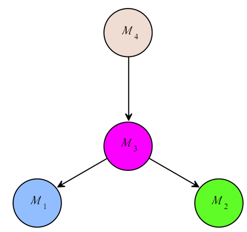

Figure 2.16: A Morse decomposition of combinatorial Lorenz system.

The families with more than one element obtained from Kosaraju's algorithm shown in (2.20) are SCC and they correspond to Morse sets which coincides with what we found in previous section. Therefore, Figure 2.16 is the minimal Morse decomposition for the model in Figure 2.1.

Next we use Kosaraju's algorithm to detect the SSC's of the model in Figure 2.6. Applying the Kosaraju's algorithm to the digraph of the model in Figure 2.6 without computer assistance will be very time consuming. To reduce the complexity of the problem, we remove the index 1 loops from the model, shown in Figure 2.17, and then we apply the Kosaraju's algorithm to the digraph of the remaining model, shown in Figure 2.18. The union of index 1 loops is already a strongly connected component.

In the first stage of Kosaraju's algorithm, we apply a DFS on the digraph in Figure 2.17 which results in the labeled digraph in Figure 2.19. The initial node to start the DFS algorithm is the node associated with the σ_1. The starting and finishing time corresponding to each node is shown as a pair beside it.

In the second stage of Kosaraju's algorithm, the arrows in the labeled digraph obtained from previous stage should be reversed. This results in the labeled digraph in Figure 2.20.

Now, the nodes of the Figure 2.20 should be taking into a stack in decreasing order with respect to finishing time. We taking the nodes one by one from stack and do a DFS starting from that node to discover the SCC's one by one. At each step the nodes of the discovered SSC should be removed from the stack. This procedure should be done until stack is empty. This results in the following

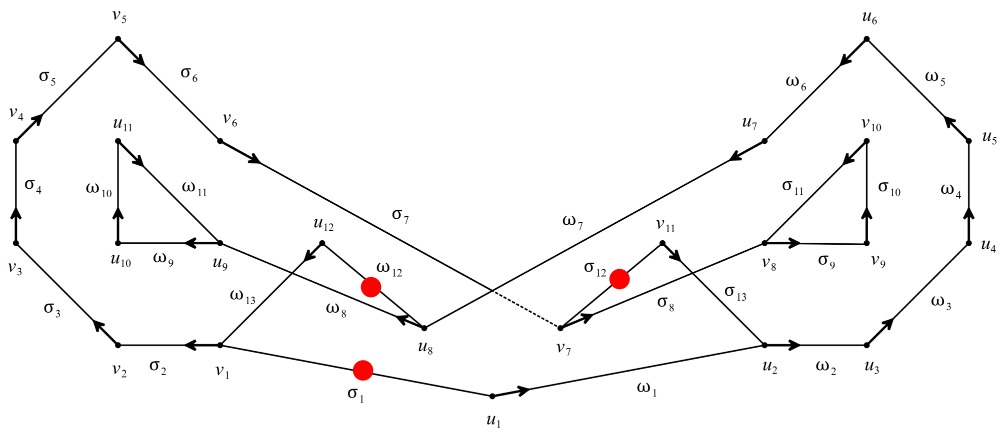

Figure 2.17: The combinatorial dynamical model in Figure 2.6 with index 1 loops removed.

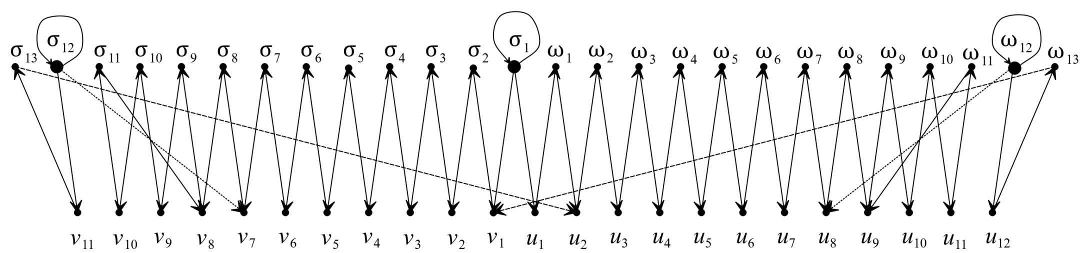

Figure 2.18: The digraph associated with the combinatorial dynamical system shown in Figure 2.17

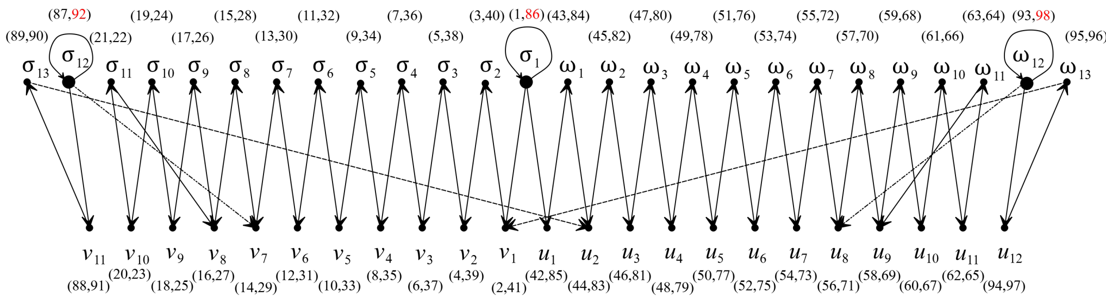

Figure 2.19: Kosaraju's algorithm (first stage): The output of DFS applied to digraph in Figure 2.18

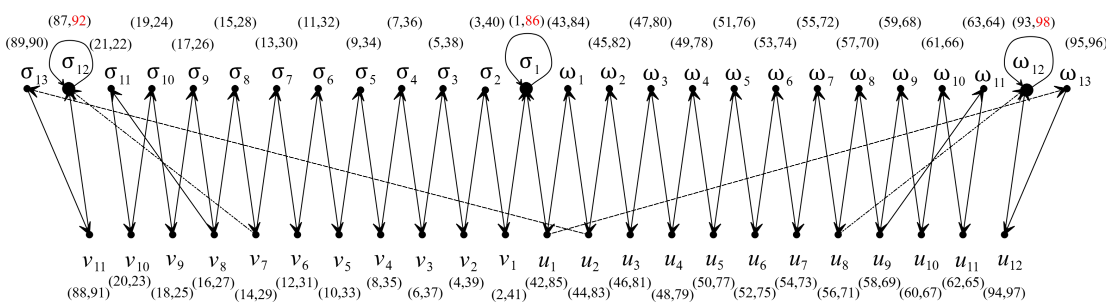

Figure 2.20: Kosaraju's algorithm (second stage): The arrows in digraph Figure 2.19 get reversed.

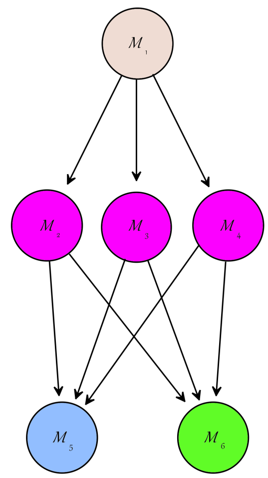

Figure 2.21: A Morse decomposition for the modified combinatorial Lorenz system in simplicial complex setting. The light brown, purple, green, blue colors demonstrate the Morse sets.

SCC's:

$$\begin{aligned}&\{\omega_{12},\omega_{12}\},\{u_{12}\},\{\omega_{13}\},\{\sigma_{12},\sigma_{12}\},\{v_{11}\},\{\sigma_{13}\},\{\sigma_1,\sigma_1\},\{u_1\},\{\omega_1\},\\&\{u_2\},\{\omega_2\},\{u_3\},\{\omega_3\},\{u_4\},\{\omega_4\},\{u_5\},\{\omega_5\},\{u_6\},\{\omega_6\},\{u_7\},\{\omega_7\},\\&\{u_8\},\{\omega_8\},\{u_9,\omega_{11},u_{11},\omega_{10},u_{10},\omega_9\},\{v_1\},\{\sigma_2\},\{v_2\},\{\sigma_3\},\{v_3\},\{\sigma_4\},\\&\{v_4\},\{\sigma_5\},\{v_5\},\{\sigma_6\},\{v_6\},\{\sigma_7\},\{v_7\},\{\sigma_8\},\{v_8,\sigma_{11},v_{10},\sigma_{10},v_9,\sigma_9\},\end{aligned} \tag{2.21}$$

The families in (2.21) with at least two elements are

$$\begin{aligned}&\mathcal{M}_2=\{\omega_{12},\omega_{12}\},\ \mathcal{M}_3=\{\sigma_{12},\sigma_{12}\},\ \mathcal{M}_4=\{\sigma_1,\sigma_1\},\\&\mathcal{M}_5=\{u_9,\omega_{11},u_{11},\omega_{10},u_{10},\omega_9\},\ \mathcal{M}_6=\{v_8,\sigma_{11},v_{10},\sigma_{10},v_9,\sigma_9\}\end{aligned}$$

which are the strongly connected components of the digraph in Figure 2.17. The strongly connected components detected in (2.21) coincide with what we found in previous section. Therefore, Figure 2.21 is a minimal Morse decomposition for the model in Figure 2.6.

Chapter 3

Toward Conley Index Theory on Cubical Complexes: Multi-Valued Map Approach

Vast studies on Conley index theory on combinatorial dynamical systems in simplicial setting have been done. The multi-valued map approach can be found in [13]. The supplementary studies toward index pair, weak index pair and Conley index theory is pursued in [2], [4], [3], [15]. The studies of Conley index is generalized to Lefschetz complexes in [20]. In this chapter, we follow similar approach done in [13] toward Conley index theory of cubical complexes, which are a particular type of Lefschetz complexes.

3.1 Cubical Complex

The formal topological and combinatorial definitions of a cubical complex are introduced in [16]. We recall the basics to this notion.

An *elementary interval* is closed interval $I \subset \mathbb{R}$ of the form $[l, l+1]$ or $[l, l]$ for some $l \in \mathbb{Z}$. For simplicity, $[l, l]$ is denoted by $[l]$. For every integer $l \in \mathbb{Z}$, the elementary intervals of the form $[l, l]$ is *degenerate* and those of the form $[l, l+1]$ are *nondegenerate*.

An *elementary cube* C is a finite product of elementary intervals, which is

$$C = I_1 \times I_2 \times \ldots \times I_d \subset \mathbb{R}^d \tag{3.1}$$

where the intervals I_i are elementary intervals. The set of all elementary cubes in $\mathbb{R}^d$ is denoted by K^d. The set of all elementary cubes is denoted by K, namely

$$K = \bigcup_{d=1}^{\infty} K^d. \tag{3.2}$$

For a cube $C = I_1 \times I_2 \times \ldots \times I_k \subset \mathbb{R}^k$, the *embedding number* of C denoted by emb(C) is k, and the *dimension* of C is the number of non-degenerate components of C, and it is denoted by dim(C). The relation between embedding number of a cube C and its dimension is

$$0 \leq \text{dim}(C) \leq \text{emb}(C). \tag{3.3}$$

The set of elementary cubes of dimension d is denoted by Ω_d and is defined as

$$K_d = \{C \in K \mid \text{dim}(C) = d\}. \tag{3.4}$$

The set of elementary cubes with dimension d and embedding number k is denoted by K_d^k and is equal to

$$K_d^k = K_d \cap K^k. \tag{3.5}$$

Let

$$C_1 = I_1 \times I_2 \times \ldots \times I_{k_1} \in K_{d_1}^{k_1} \text{ and } C_2 = J_1 \times J_2 \times \ldots \times J_{k_2} \in K_{d_2}^{k_2}, \tag{3.6}$$

then, product of elementary cubes $C_1 \times C_2$ is defined as

$$C_1 \times C_2 := I_1 \times I_2 \times \ldots \times I_{k_1} \times J_1 \times J_2 \times \ldots \times J_{k_2}. \tag{3.7}$$

Clearly, $C_1 \times C_2 \in K_{d_1+d_2}^{k_1+k_2}$.

Let $C_1, C_2 \in K$. If $C_2 \subseteq C_1$ then C_2 is called a *face* of C_1. If $C_2 \subseteq C_1$ but $C_2 \neq C_1$, then C_2 is called a *proper face* of C_1. The elementary cube $C_2 \subseteq C_1$ is called a *primary face* or a *facet* of C_1, if dim$(C_2) = $ dim$(C_1) - 1$.

A set $X \subset \mathbb{R}^d$ is *cubical*, if X can be written as a finite union of elementary cubes. The set of elementary cubes of X is defined as

$$K(X) = \{C \in K \mid C \subseteq X\}. \tag{3.8}$$

The set of elementary cubes in X of dimension d, or d*-cubes* of X is defined as

$$K_d(X) = \{C \in K(X) \mid \text{dim}(C) = d\}. \tag{3.9}$$

The set of elementary cubes in X with embedding number k is defined as

$$K^k(X) = \{C \in K(X) \mid \text{emb}(C) = k\}. \tag{3.10}$$

The elementary cubes of embedding number k and dimension d is $K_d(X) \cap K^k(X)$ and is denoted by $K^k_d(X)$. Elements of $K_0(X)$ and $K_1(X)$ are called *vertices* of X and *edges* of X, respectively.

A non-empty set $\mathcal{K}$ of elementary cubicals is called a *cubical complex* if $\mathcal{K} = K(X)$ for some elementary cube X.

Assume $\mathcal{K}$ is a finite cubical complex and let $k = \max\{\text{emb}(C) \mid C \in \mathcal{K}\}$. Then $\mathcal{K}$ can be embedded into $\mathbb{R}^k$ and every cubical $C = I_1 \times \cdots \times I_k \in \mathcal{K}$ of dimension l can be alternatively presented in the form $[v_1, \ldots, v_{2^k}]$ where $v_1, \ldots, v_{2^k}$ are vertices of $C \subset \mathbb{R}^k$. For the purpose of cell decomposition we will need to consider cubes or rectangles not necessarily with integer coordinate vertices.

The set of all faces of $C \in \mathcal{K}$ is called the *combinatorial closure* of C and it is denoted by Cl(C) and the *topological closure* of C is the closure of C as a subset of $\mathbb{R}^d$ and it is denoted by cl(C). The *combinatorial boundary* of a cube C denoted by Bd(C) is the defined as the set of all proper faces of the cubical C.

3.2 A Substitution for Barycentric Coordinates in Cubical Setting

The barycentric coordinates of a point in a simplicial complex play an essential role in the construction and development of the multi-valued map approach toward

Conley index theory done in [13]. For a simplex $\sigma = [v_0, v_1, \ldots, v_n]$ and $x \in |\sigma|$, the point x can be written as

$$x = \sum_{i=0}^{n} t_{v_i}(x) v_i \tag{3.11}$$

such that

(i) $0 \leq t_{v_i} \leq 1, \; i = 0, \ldots, n.$

(ii) $\sum_{i=0}^{n} t_{v_i}(x) = 1.$

The coefficient $t_{v_i}(x)$ in Equation 3.11 is called the barycentric coordinate of x with respect to vertex v_i.

In the cubical setting, as apposed to the simplicial setting, since the vertices associated with a cube, when considered as points in the Euclidean space are no longer colinear, the notion of barycentric coordinates will be meaningless. Lack of not having barycentric coordinates in the cubical setting is an obstacle to develop the results toward Conley index theory on simplicial complex, for cubical setting. In this chapter we replace the barycentric coordinate with a new function which plays similar role. That function is in fact a particular metric defined on Euclidean spaces which we will discuss later in this chapter. First, we see some attempts done to compensate the lack of barycentric coordinates in the cubical setting for transferring notions defined from simplicial setting to cubical setting. In this chapter we consider unit cubes unless otherwise stated.

For simplicity, we start working with the cubical complex of dimension 2.

Let $\mathcal{X}$ denote a finite cubical complex, $\mathcal{V}$ denote a combinatorial vector field defined on $\mathcal{X}$. Moreover, let $X = |\mathcal{X}|$ be the geometrical realization of $\mathcal{X}$ as a subset of $\mathbb{R}^2$.

Assume $\sigma = [e_1, e_2, e_3, e_4]$ be a cube and $x \in \mathbb{R}^2$, where e_i's denote the edges of σ. Let $d_{e_i}(x)$ denote the shortest distances of x from edge e_i defined as following:

$$d_e(x) = \begin{cases} \text{Shortest distance of } x \text{ from } e & \text{if } e \in \sigma \text{ and } x \in |\sigma|, \\ \infty & \text{otherwise.} \end{cases} \tag{3.12}$$

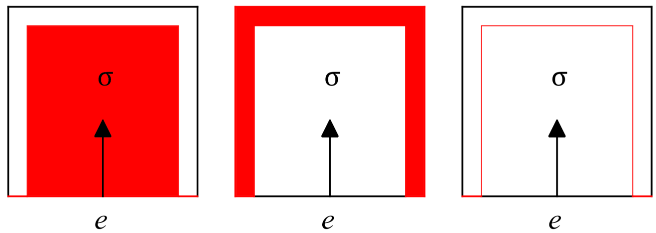

Figure 3.1: A_e, B_e, and C_e sets with respect to metric d_e

Clearly, for every $x \in |\sigma|$ we have $\sum\limits_{e\in\sigma} d_e(x) = 2$.

Having this metric defined on a cube, now we try to construct the objects created in [13] in order to construct the multi-valued map associated with a combinatorial vector field. Here, the metric d_e is supposed to play the role of barycentric function.

For a parameter $0 < \lambda < 1/2$ and edge e of cube σ we define

$$\begin{aligned} A_e &:= \{x \in |\sigma| \mid d_e(x) \leq 1-\lambda \text{ and } \forall_{f\in e^{\perp}\cap\sigma}\, d_f(x) \leq 1-\lambda\} \cup |e|, \\ B_e &:= \{x \in |\sigma| \mid d_e(x) \geq 1-\lambda \text{ or } \exists_{f\in e^{\perp}\cap\sigma}\, d_f(x) \geq 1-\lambda\}, \\ C_e &:= A_e \cap B_e. \end{aligned}$$

Where $e^{\perp}$ is the set of all edges in σ which are geometrically perpendicular to e.

In Figure 3.1 we can see how the sets A_e, B_e and C_e look like.

Moreover, we define λ-*characteristic set* denoted by $\mathcal{X}^{\lambda}(x)$ to be

$$\mathcal{X}^{\lambda}(x) := \Big\{\sigma \;\Big|\; \forall_{v\in\sigma}\, \{\forall_{e;v\in e}, d_e(x) \leq 1-\lambda\}; \text{and } \forall_{v\notin\sigma}\{\exists_{e;v\in e}, d_e(x) \geq 1-\lambda\}\Big\} \tag{3.13}$$

and the *minimum* and *maximum characteristic cube* to be

$$c_m^{\lambda}(x) := \Big[v \in \mathcal{X}_0 \;\Big|\; \forall_{e\in\mathcal{X}_1;v\in e},\; d_e(x) < 1-\lambda\Big] \tag{3.14}$$

$$c_M^{\lambda}(x) := \Big[v \in \mathcal{X}_0 \;\Big|\; \forall_{e\in\mathcal{X}_1,v\in e},\; d_e(x) \leq 1-\lambda\Big] \tag{3.15}$$

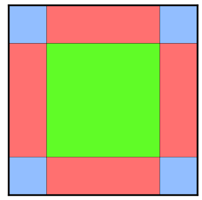

Figure 3.2: Cell decomposition of a 2-dimensional cube with respect to metric d_e

and finally the λ-cell of a cube σ is defined as

$$\langle\sigma\rangle_\lambda := \Big\{x \in X \mid \forall_{v\in\sigma}\ \{\forall_{e;v\in e}, d_e(x) \leq 1-\lambda\}; \text{and } \forall_{v\notin\sigma}\{\exists_{e;v\in e}, d_e(x) \geq 1-\lambda\}\Big\}. \tag{3.16}$$

In Figure 3.2 we can observe the cell decomposition of a cube into mutually disjoint cells. The green, red, blue region associated with the cell of the cube of dimension 2, 1, and 0 respectively.

Even though for a cube σ of dimension at most 2, the A_σ, B_σ and C_σ sets as well as $c_m(x)$, $c_M(x)$ and the λ-cell of σ are geometrically analogous to those notions in simplicial case, this approach is not very convenient to be developed to higher dimensions. More precisely, for definition of the sets A_σ, B_σ and C_σ when σ has dimension 3 or higher, the metric d_e is not useful anymore as its definition needs to be modified according to dimension of the cube. Therefore, the metric d_e is not universal and works only for dimension 2.

Still, we assume that cubical complexes have dimension at most 2. The metric d_e in preceding argument showed not to be a good candidate to play the role of barycentric coordinates in cubical setting. Toward transferring the Conley index theory, the notions, and theorems of simplicial complexes into cubical setting we look for another function similar to barycentric function which is uniquely defined for every point of simplex, and the barycentric functions values on a point x can uniquely locate the point in the simplicial complex. Moreover, we note that barycentric function associates a quantity to a point in geometric realization of simplex and each of the simplex vertices. In fact the failure in first attempt originated from the fact that the metric $d_e(x)$ does not associate x with a vertex, instead associates x with and edge. Therefore, in higher dimension we have to replaced metric d_e with another metric. That can make it very difficult to transfer definitions.

As the next effort, we try different metrics on Euclidean space and verify if that would appropriately transfer the definitions from simplicial setting to cubical setting. Among the several commonly used metrics on Euclidean space, we observe one works as we expect. In next section we discuss that further.

3.3 Cell Decomposition of a Cubical Complex

In this section we test different metrics on Euclidean space for obtaining a similar cell decomposition of simplicial complex for a cubical complex.

The definition of λ-cell of a simplex σ was defined to be

$$\langle\sigma\rangle_\lambda = \{x \in X \mid t_v(x) > \lambda \text{ for all } v \in \sigma, \text{ and } t_v(x) < \lambda \text{ for all } v \notin \sigma\}. \tag{3.17}$$

Assume σ is a simplex, $v \in |\sigma| \cap \mathfrak{X}_0$ and $x \in |\sigma|$. Assume t_v to be the barycentric function with respect to vertex v. We know that $t_v(x) \in [0,1]$, moreover when $t_v(x)$ gets closer to 1, then x get closer to v and when $t_v(x)$ gets closer to 0, x gets farther from v. For a metric d on Euclidean space, a fixed vertex v of a cube σ, and $x \in |\sigma|$, when $d_v(x) := d(v,x)$ increases, x gets farther away from v, and when $d_v(x) := d(v,x)$ decreases, x gets closer to v. Therefore, to obtain the definition of λ-cell for a cube in terms of metric d_v, we should dualize the definition of λ-cell for a simplex given in Equation 3.17. By dualization of Equation 3.17 we obtain

$$\langle\sigma\rangle_\lambda = \{x \in X \mid d_v(x) < 1-\lambda \text{ for all } v \in \sigma, \text{ and } d_v(x) > 1-\lambda \text{ for all } v \notin \sigma\}. \tag{3.18}$$

Now, we depict the λ-cell associated with a cube with respect to common metrics on Euclidean space.

One can consider the following equivalent norms on the Euclidean space $\mathbb{R}^n$:

(i) $||(x_1, \cdots, x_n)||_1 = |x_1| + \cdots + |x_n|$,

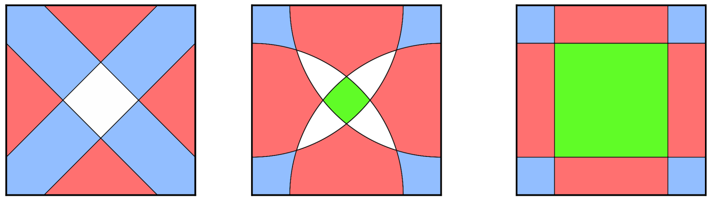

Figure 3.3: From left to right, the λ-cell decomposition of σ with respect the three common norms $\|.\|_1$, $\|.\|_2$ and $\|.\|_\infty$ on Euclidean space $\mathbb{R}^n$, respectively.

(ii) $||(x_1, \cdots, x_n)||_2 = \sqrt{{x_1}^2 + \cdots + {x_n}^2}$,

(iii) $||(x_1, \cdots, x_n)||_\infty = \max_{1 \leq i \leq n} |x_i|$.

Let σ be a cube of dimension 2. The figure of the λ-cell of σ with respect to the above metrics and Definition 3.18 are shown in Figure 3.3

λ-cell decomposition of σ with respect to $\|.\|_1$ is not acceptable because the λ-cell associated with the cube of dimension 2 is empty. Also, λ-cell decomposition of σ with respect to $\|.\|_2$ is not acceptable since there are regions in $|\sigma|$ which does not belong to any of the cubes cell. But λ-cell decomposition of σ with respect to $\|.\|_\infty$ is exactly what we were attempting to obtain and it is analogous to λ-cell decomposition on simplices. Having this success we continue the construction of rest of the essential notions toward Conley index in terms of metric $\|.\|_\infty$.

3.4 λ-Characteristic Cubes, Minimal/Maximal λ- Characteristic Cubes

For the rest of this chapter, the metric d is induced by the infinity norm $\|.\|_\infty$ on $\mathbb{R}^n$ unless otherwise specified.

Let $\mathcal{X}$ be a cubical complex of dimension n and let $X = |\mathcal{X}|$ denote the geometric realization of $\mathcal{X}$. Fix $0 \leq \lambda < \frac{1}{2}$. For every $x \in X$ define the λ-*signature function* of x as

$$\begin{aligned} \mathrm{sign}^\lambda x : \mathcal{X}_0 &\longrightarrow \{-1,0,1\} \\ v &\longmapsto \mathrm{sgn}\big((1-\lambda) - d_v(x)\big), \end{aligned}$$

where sgn : $\mathbb{R} \longrightarrow \{-1,0,1\}$ is the standard sign function.

For $x \in X$ and $v \in \mathcal{X}_0$ define the metric d_v on $|\sigma|$ to be

$$d_v(x) = \begin{cases} d(v,x) & \text{if } \exists \sigma \in \mathcal{X},\ v \in \sigma, x \in |\sigma|, \\ \infty & \text{otherwise.} \end{cases} \tag{3.19}$$

A cube σ is called a λ-characteristic cube of x, if both $\mathrm{sign}^\lambda x|_\sigma \geq 0$ and $(\mathrm{sign}^\lambda x)^{-1}(\{1\}) \subseteq \sigma$ hold.

The set of all λ-*characteristic cubes* of x is denoted by $\mathcal{X}^\lambda(x)$ and is defined to be

$$\mathcal{X}^\lambda(x) := \{\sigma \in \mathcal{X} \mid (\mathrm{sign}^\lambda x)^{-1}(\{1\}) \subseteq \sigma, \text{ and } \mathrm{sign}^\lambda x(v) \geq 0 \text{ for all } v \in \sigma\} \tag{3.20}$$

The practical form of definition of $\mathcal{X}^\lambda(x)$ in terms of metric d is given by

$$\mathcal{X}^\lambda(x) := \{\sigma \in \mathcal{X} \mid d_v(x) \leq 1-\lambda \text{ for all } v \in \sigma, \text{ and } d_v(x) \geq 1-\lambda \text{ for all } v \notin \sigma\}. \tag{3.21}$$

For $1 > \lambda$, $(\mathrm{sign}^\lambda x)^{-1}(\{1\})$ is a particular cube which is called *minimal characteristic cube* of x. In terms of metric d, it is defined as

$$c_m^\lambda(x) := \{v \in \mathcal{X}_0 \mid d_v(x) < 1-\lambda\}.$$

For $1 \geq \lambda$, $(\mathrm{sign}^\lambda x)^{-1}(\{0,1\})$ is a cube which is called as *maximal characteristic cube* of x. In terms of metric d, it is defined as

$$c_M^\lambda(x) := \{v \in \mathcal{X}_0 \mid d_v(x) \leq 1-\lambda\}.$$

In Figure 3.4, a cube of dimension 2 and some sample points from it are depicted. The next table shows the corresponding minimal and maximal λ-characteristic cubes to each of those sample points.

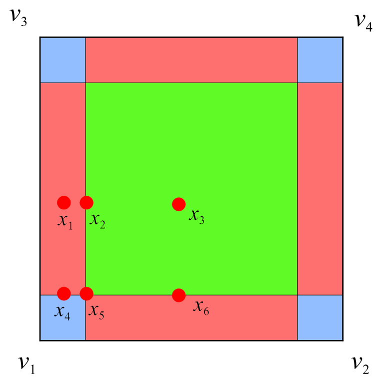

Figure 3.4: Minimal and maximal λ-characteristic cubes for certain points on a 2 dimension cube.

n	1	2	3	4	5	6
$c_m^\lambda(x_n)$	$[v_1, v_3]$	$[v_1, v_3]$	$[v_1, v_2, v_3, v_4]$	$[v_1]$	$[v_1]$	$[v_1, v_2]$
$c_M^\lambda(x_n)$	$[v_1, v_3]$	$[v_1, v_2, v_3, v_4]$	$[v_1, v_2, v_3, v_4]$	$[v_1, v_3]$	$[v_1, v_2, v_3, v_4]$	$[v_1, v_2, v_3, v_4]$

Neighborhoods and isolating neighborhoods are building blocks in development of Conley index theory in combinatorial setting. In order to define those notions in combinatorial setting, the geometric realization of a cube is needed to be broken down into smaller pieces. Each of those pieces is called a cube cell. The cell decomposition of a cube depends on a variable λ and it is defined to be

$$\langle\sigma\rangle_\lambda := \{x \in X \mid \forall v \in \sigma,\ d_v(x) < 1-\lambda;\ \forall v \notin \sigma,\ d_v(x) > 1-\lambda\}.$$

For a cube $\sigma \in \mathcal{X}$ of dimension n, the set of λ-cells of faces of σ consists of mutually disjoint open rectangles in $\mathbb{R}^n$ such that union of their closure equals $|\sigma|$. In Figure 3.5 the λ-cell decompositions of cubes of dimensions up to 3 are shown.

In construction of neighborhoods and isolating neighborhoods, the closure of λ-cells have to be used. Following shows the formula for cl $\langle\sigma\rangle_\lambda$.

$$\begin{aligned} \mathrm{cl}\langle\sigma\rangle_\lambda &= \mathrm{cl}(\{x \in X \mid \forall v \in \sigma,\ d_v(x) < 1-\lambda;\ \forall v \notin \sigma,\ d_v(x) > 1-\lambda\}) \\ &= \{x \in X \mid \forall v \in \sigma,\ d_v(x) \le 1-\lambda;\ \forall v \notin \sigma,\ d_v(x) \ge 1-\lambda\}. \quad (3.22) \end{aligned}$$

We note that the second equality in Equation 3.22 is true by the continuity of metric d_v, which is verified in more details next.

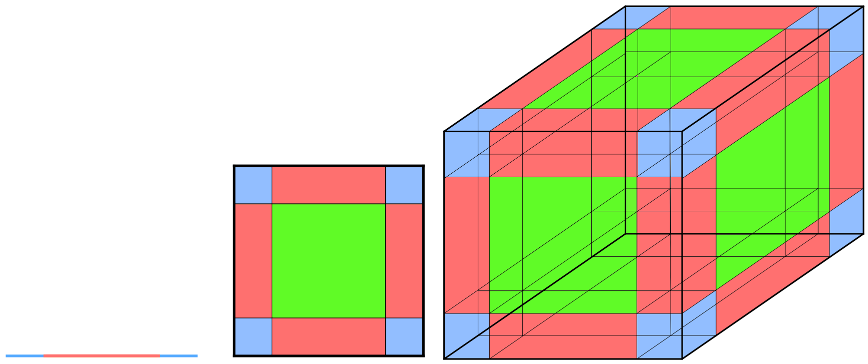

Figure 3.5: λ-cell decomposition of cubes of dimension up to 3.

Lemma 3.1. *Let $v \in \mathcal{X}_0$ and define $d_v : X \to \mathbb{R}$ as $d_v(x) = \max_{1\le i\le n} |x_i - v_i|$ then d_v is a continuous map.*

Proof. For $x, y \in X$,

$$\begin{aligned}|d_v(x) - d_v(y)| &= \Big| \max_{1\le i\le n} |x_i - v_i| - \max_{1\le i\le n} |y_i - v_i| \Big| \\ &\le \Big| \max_{1\le i\le n} (|x_i - y_i|) \Big| \\ &= d(x, y).\end{aligned}$$

In the above proof, from first line to the second, the reverse triangle inequality is used. Continuity of d_v follows.

□

Lemma 3.2. *For every $\sigma, \tau \in \mathcal{X}$, the set $\langle\sigma\rangle_\lambda \cap |\tau|$ is either empty or it is a rectangle.*

Proof. for $v \in \mathcal{X}_0$ define the following sets,

$$A_v := \{x \in X \mid d_v(x) < 1 - \lambda\},$$
$$B_v := \{x \in X \mid d_v(x) > 1 - \lambda \text{ and } d_v(x) < \infty\}$$

By definition,

$$\begin{aligned}\langle\sigma\rangle_\lambda &= \{x \in X \mid \forall v \in \sigma,\ d_v(x) < 1 - \lambda;\ \forall v \notin \sigma,\ d_v(x) > 1 - \lambda\} \\ &= \bigcap_{v\in\sigma} A_v \cap \bigcap_{v\notin\sigma} B_v. \end{aligned} \tag{3.23}$$

Assume $\sigma = [v_1, \cdots, v_{2^n}]$ is a facet of cube $\tau = [v_1, \cdots, v_{2^n}, u_1, \cdots, u_{2^n}]$. Consider the origin to be located at the centroid of $|\sigma|$ with an orthogonal coordinate system defined on $|\sigma|$ and extended to an orthogonal coordinate system on $|\tau|$. Then $\bigcap\limits_{v\in\sigma} A_v$ will be of the form

$$\big(-(\frac{1}{2}-\lambda),(\frac{1}{2}-\lambda)\big)\times\cdots\times\big(-(\frac{1}{2}-\lambda),(\frac{1}{2}-\lambda)\big)\times\big(0,1-\lambda)\times\cdots\times(0,1-\lambda\big).$$

On the other hand, $\bigcap\limits_{v\notin\sigma} B_v$ will be of the form,

$$(-\frac{1}{2},\frac{1}{2})\times\cdots\times(-\frac{1}{2},\frac{1}{2})\times(0,\lambda)\times\cdots\times(0,\lambda)$$

Therefore, $\bigcap\limits_{v\in\sigma} A_v \cap \bigcap\limits_{v\notin\sigma} B_v$ forms a cube in $\mathbb{R}^{n+1}$.

We showed that the assertion is true for every facet of of τ. Analogous argument can be made to show the validity of assertion for every face of τ. $\square$

Lemma 3.3. *let $\mathcal{X}$ denote a cubical complex in $\mathbb{R}^d$. Then for every cube $\sigma \in \mathcal{X}$ the associated λ-cell $\langle\sigma\rangle_\lambda$ is a non-empty open set in X, and the intersection of σ with any cube of $\mathcal{X}$ is convex. Furthermore, the closure of the λ-cell can be expressed in terms of metric d_v as*

$$cl\langle\sigma\rangle_\lambda = \{x \in X \mid \forall v \in \sigma,\ d_v(x) \le 1-\lambda;\ \forall v \notin \sigma,\ d_v(x) \ge 1-\lambda\}. \quad (3.24)$$

Proof. For any cubical complex σ, its centroid belongs to $\langle\sigma\rangle_\lambda$, hence $\langle\sigma\rangle_\lambda$ is a non-empty set. Moreover, $\langle\sigma\rangle_\lambda = \bigcap_{v\in\sigma} d_v^{-1}((0,1-\lambda)) \cap \bigcap_{v\notin\sigma} d_v^{-1}((1-\lambda,\sqrt{d}))$ where d is the dimension of cube σ, also the barycenteric functions are continuous, so $\langle\sigma\rangle_\lambda$ is open.

Then,

$$\langle\sigma\rangle_\lambda = \bigcap\nolimits_{v\in\sigma} A_v \cap \bigcap\nolimits_{v\notin\sigma} B_v\ .$$

For an arbitrary cube $\tau \in \mathcal{X}$, and $v \in \mathcal{X}_0$, τ, A_v and B_v are convex and so are $|\tau| \cap A_v$ and $|\tau| \cap B_v$, hence $\langle\sigma\rangle_\lambda \cap |\tau|$ is convex.

Note that closure of intersection of two sets is not always equal to intersection of closures. So it is not trivial to conclude the claim for cl $\langle\sigma\rangle_\lambda$ right away.

Suppose $x \in \text{cl}(\langle\sigma\rangle_\lambda)$. Then by definition, there is a sequence $\{x_n\}_{n\ge1} \subset \langle\sigma\rangle_\lambda$ which converges to x. That means for all n, $d_v(x_n) < 1-\lambda$ for all $v \in \sigma$ and $d_v(x_n) > 1-\lambda$ for all $v \notin \sigma$. Thus, x is contained in the right-hand side of

Equation 3.24.

Conversely, assume $X \subset \mathbb{R}^n$ and $x = (x_1, \ldots, x_n)$ belongs to the right hand side of Equation 3.24 such that at least for one $v \in \mathcal{X}_0$, $d_v(x) = 1 - \lambda$, otherwise we get $x \in \langle\sigma\rangle_\lambda \subset \mathrm{cl}(\langle\sigma\rangle_\lambda)$. Define the following sets

$$\begin{array}{llll} v \in E_1 & \text{if and only if} & d_v(x) = 1 - \lambda & \text{and} \quad v \in \sigma, \\ v \in E_2 & \text{if and only if} & d_v(x) = 1 - \lambda & \text{and} \quad v \notin \sigma, \\ v \in E_3 & \text{if and only if} & d_v(x) \neq 1 - \lambda, & \end{array}$$

also let $(\delta_1, \ldots, \delta_n) \in \mathbb{R}$ where $\delta_i \in \{-\delta, 0, \delta\}$, $i = 1, \ldots, n$. Then for sufficiently small $\delta > 0$ and an appropriate choice of $(\delta_1, \ldots, \delta_n)$, we can insure that $x_\delta = (x_1 + \delta_1, \ldots, x_n + \delta_n)$, we have $d_v(x_\delta) < 1 - \lambda$ for all $v \in \sigma$ and $d_v(x_\delta) > 1 - \lambda$ for all $v \notin \sigma$. Therefore $x_\delta \in \langle\sigma\rangle_\lambda$. But $x_\delta \to x$ as $\delta \to 0$ where, $x_\delta \in \langle\sigma\rangle_\lambda$ hence, $x \in \mathrm{cl}\langle\sigma\rangle_\lambda$.

□

The cube decomposition built above was independent of the vector field defined on the cubical complex. The cube decomposition which we discuss later in this section will be based on the behavior of vector field defined on the cubical complex. We note that the combinatorial vector field on a cubical complex is defined in the same way as on a simplicial complex given in Definition 2.3. It still worth to recall this definition in cubical setting here:

Definition 3.4. *A map $f : X \nrightarrow Y$, is called partial map if its domain is a subset of X. We denote image of f and fixed points of f by Im$(f) = f(X)$ and Fix$(f) = \{x \in$ Dom$(f) \mid f(x) = x\}$ respectively.*

Definition 3.5. *An injective partial self-map $\mathcal{V} : \mathcal{X} \nrightarrow \mathcal{X}$ of a cubical complex $\mathcal{X}$ is called a combinatorial or discrete vector field, if the following conditions hold:*

(i) $\forall \sigma \in$ Dom$(\mathcal{V})$ either $\mathcal{V}(\sigma) = \sigma$, or σ is a facet of $\mathcal{V}(\sigma)$,

(ii) Dom$(\mathcal{V}) \bigcup$ Im$(\mathcal{V}) = \mathcal{X}$,

(iii) Dom$(\mathcal{V}) \bigcap$ Im$(\mathcal{V}) =$ Fix$(\mathcal{X})$.

A combinatorial vector field partitions $\mathcal{X}$ into mutually disjoint subsets of form $\{\sigma, \mathcal{V}(\sigma)\}$ such that either $\mathcal{V}(\sigma) = \sigma$, or σ is a facet of $\mathcal{V}(\sigma)$. One can demonstrate this pairing on X by arrows and red dots. For a set $\{\sigma, \mathcal{V}(\sigma)\}$ with $\mathcal{V}(\sigma) \neq \sigma$, on X the arrow tail initiates from σ and ends in $\mathcal{V}(\sigma)$, and if $\mathcal{V}(\sigma) = \sigma$, then the cube σ will be marked with a red dot standing for a critical point of the vector field.

The arrows and critical cells of vector field in X tell us about how the flow in background continuous dynamic behaves. For critical points marked by red dots, depending on the dimension of the cube, one can tell in how many directions, flow repells from or attracts to the critical point.

Moreover, the arrows of vector field on X could be also used for elementary collapses of $\mathcal{V}(\sigma)$ onto $\mathrm{bd}(|\mathcal{V}(\sigma)|)\backslash|\sigma|$, when σ is only face of one cube. That could be used as a tool to reduce the complexity in geometrical representation of the dynamic on $\mathcal{X}$.

In order to study deeper the dynamic of a combinatorial vector field $\mathcal{V}$ defined on a finite cubical complex $\mathcal{X}$, and to have a better tie between the combinatorial dynamic and the underlaying time continuous dynamic, one approach is to associate a multi-valued map F to $\mathcal{V}$. The orbits and isolated invariant sets of F and the flow associated with $\mathcal{V}$ denoted by $\Pi_{\mathcal{V}}$ are in fact in a one to one correspondence. That approach was initiated and developed for finite simplicial complexes in [13].

Building blocks of multi-valued self map $F : X \to X$ which is supposed to exhibit the background dynamic of the combinatorial vector field, are the particular subdivision of $|\sigma|$ into three non-empty, acyclic, compact sets which are defined precisely next.

Let $\mathcal{V}$ be a combinatorial vector field defined on a finite cubical complex $\mathcal{X}$. Moreover assume $\sigma \in \mathcal{X}$. For convenience in discussions and proofs, the source/tail and target/head of an arrow is denoted with the following notations.

$$\sigma^+ := \begin{cases} \mathcal{V}(\sigma) & \text{if } \sigma \in \mathrm{Dom}(\mathcal{V}), \\ \sigma & \text{otherwise} \end{cases} \quad \text{and} \quad \sigma^- := \begin{cases} \sigma & \text{if } \sigma \in \mathrm{Dom}(\mathcal{V}), \\ \mathcal{V}^{-1}(\sigma) & \text{otherwise.} \end{cases} \tag{3.25}$$

It is easy to see but important in next proofs to pay attention that for a cube inclusion $\sigma \subset \tau$, one can neither conclude $\sigma^+ \subset \tau^+$ nor $\sigma^- \subset \tau^-$. An example is shown in Figure 3.6.

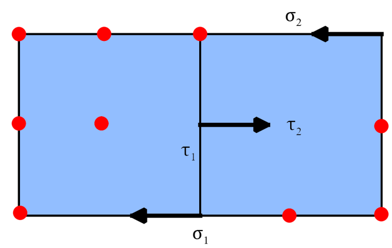

Figure 3.6: In this cubical complex, the inclusion $\sigma_1 \subset \tau_1$ holds while ${\sigma_1}^+ \not\subset {\tau_1}^+$. Also, the inclusion $\sigma_2 \subset \tau_2$ holds while ${\sigma_2}^- \not\subset {\tau_2}^-$.

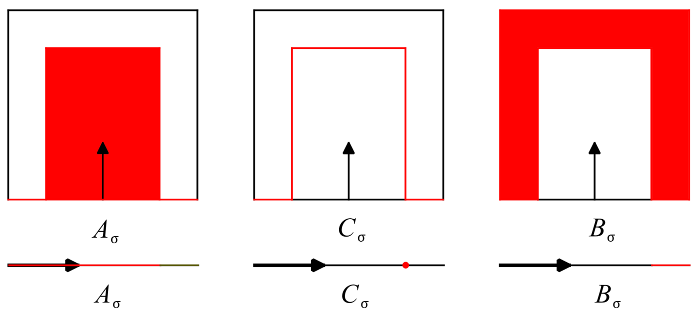

Figure 3.7: A_σ, B_σ, and C_σ set for a cube of dimension 2.

Let $\sigma \in \mathcal{X}$, we define the following particular subsets of $|\sigma|$:

$$A_\sigma := \left\{x \in \sigma^+ \mid \forall v \in \sigma^-, d_v(x) \leq 1 - \lambda\right\} \cup |\sigma^-|, \tag{3.26}$$
$$B_\sigma := \left\{x \in \sigma^+ \mid \exists v \in \sigma^-, d_v(x) \geq 1 - \lambda\right\}, \tag{3.27}$$
$$C_\sigma := A_\sigma \bigcap B_\sigma. \tag{3.28}$$

For a cube of dimension 2, the sets A_σ, B_σ, and C_σ are shown in Figure 3.7.

For every combinatorial vector field defined on a cubical complex, similar to the case of simplicial complex, one can associate a multi-valued map or flow denoted by $\Pi_\mathcal{V}$. The flow associated with $\mathcal{V}$ is required in order to fit better the combinatorial dynamic defined on $\mathcal{X}$ with the background time continuous dynamic on X. By solutions of $\mathcal{V}$ given by Robin Forman, we are not able to associate a combinatorial orbit with a background time continuous orbit which asymptotically repels from or attract to a critical point or a periodic orbit. Moreover, having the finiteness condition on cubical complex does not allow for having infinite trajectories with respect to $\mathcal{V}$. However, $\Pi_{\mathcal{V}}$ fills these two gaps, and it is defined next.

Given a combinatorial vector field $\mathcal{V}$ on $\mathcal{X}$, the associated combinatorial flow denoted by $\Pi_{\mathcal{V}} : \mathcal{X} \multimap \mathcal{X}$ is defined to be

$$\Pi_{\mathcal{V}}(\sigma) := \begin{cases} \mathrm{Cl}(\sigma) & \text{if } \sigma \in \mathrm{Fix}(\mathcal{V}), \\ \mathrm{Bd}(\sigma) \backslash \{\sigma^-\} & \text{if } \sigma \in \mathrm{im}\mathcal{V} \setminus \mathrm{Fix}(\mathcal{V}), \\ \{\sigma^+\} & \text{if } \sigma \in \mathrm{dom}(\mathcal{V}) \setminus \mathrm{Fix}(\mathcal{V}) \end{cases} \tag{3.29}$$

The flow $\Pi_{\mathcal{V}}$ is the intermediate combinatorial multi-valued map which will be used in constructing the upper semi-continuous multi-valued map F on X with acyclic values. It has been shown in [13] that there is a bijection between solutions of $\Pi_{\mathcal{V}}$ and F. Moreover, the isolated invariant sets under $\Pi_{\mathcal{V}}$ are in correspondence with isolating blocks under F.

Let $\mathcal{V}$ denote a combinatorial vector field on a finite cubical complex $\mathcal{X}$. Moreover, assume $\Pi_{\mathcal{V}}$ to be the flow associated with $\mathcal{V}$. Then for every cube $\sigma \in \mathcal{X}$, the multi-valued map $F_\sigma : X \multimap X$ associated with $\mathcal{V}$ is defined as follows. For every $\sigma \in \mathcal{X}$, and $0 < \varepsilon < \lambda < \frac{1}{2}$, define

$$F_\sigma(x) := \begin{cases} \emptyset & \text{if } \sigma \notin \mathcal{X}^\varepsilon(x), \\ A_\sigma & \text{if } \sigma \in \mathcal{X}^\varepsilon(x),\ \sigma \neq c^\varepsilon_M(x)^+,\ \textit{and}\ \sigma \neq c^\varepsilon_M(x)^-, \\ B_\sigma & \text{if } \sigma = c^\varepsilon_M(x)^+ \neq c^\varepsilon_M(x)^-, \\ C_\sigma & \text{if } \sigma = c^\varepsilon_M(x)^- \neq c^\varepsilon_M(x)^+, \\ |\sigma| & \text{if } \sigma = c^\varepsilon_M(x)^- = c^\varepsilon_M(x)^+. \end{cases} \tag{3.30}$$

The multi-valued map $F : X \multimap X$ associated with $\mathcal{V}$ is defined to be

$$F(x) := \bigcup_{\sigma \in \mathcal{X}} F_\sigma(x). \tag{3.31}$$

Lemma 3.6. *Let $\mathcal{X}$ be a cubical complex, $\mathcal{V}$ be a combinatorial vector field on $\mathcal{X}$, and let F be the multi-valued function defined as above. Then F_σ is well-defined, moreover,*

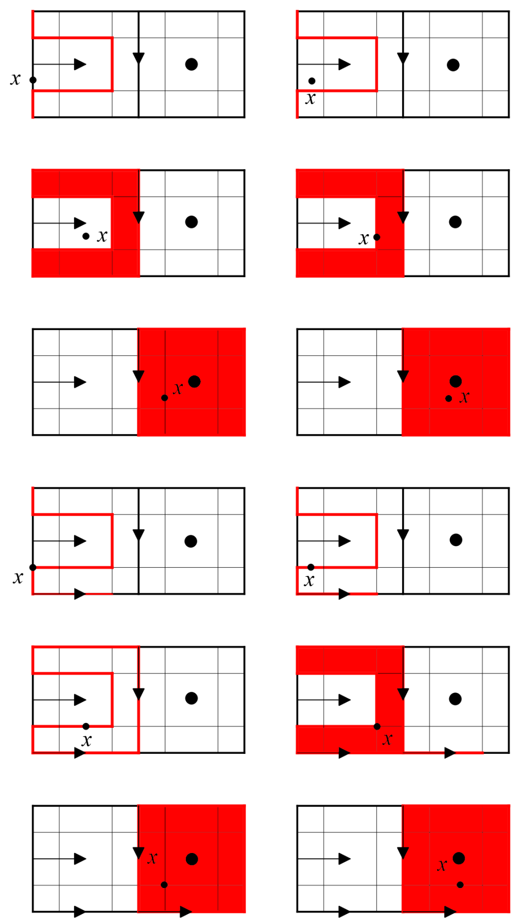

Figure 3.8: Values of the map F for different x in X.

$$F_\sigma(x) \subset \sigma^+, \textit{ for all } \sigma \in \mathcal{X} \textit{ and } x \in X.$$

Proof. We should show that F_σ defined above is exhaustive and exclusive since for every $x \in X$, one and exactly one of the five conditions will be satisfied. Now, let $\sigma \in \mathcal{X}$ and $x \in X$. By definition, the sets A_σ, B_σ, C_σ, each are subsets of σ^+. Hence $F_\sigma(x) \subset \sigma^+$ holds. □

The isolated invariant sets are the building blocks in Conley index theory of combinatorial dynamical systems. The definition of invariant set with respect to flow $\Pi_\mathcal{V}$ is analogous to its definition in simplicial case. It is defined to be:

Definition 3.7. *Let $\mathcal{V}$ be a combinatorial vector field defined on a finite cubical complex $\mathcal{X}$. Furthermore, assume $\Pi_\mathcal{V}$ is the flow associated with $\mathcal{V}$ defined in Equation 3.29. A set $\mathcal{S} \subset \mathcal{X}$ is called invariant under the flow $\Pi_\mathcal{V}$ if for every cube $\sigma \in \mathcal{S}$, there is a full solution $\varrho : \mathbb{Z} \to \mathcal{S}$ such that $\varrho(0) = \sigma$. In other words, for every $k \in \mathbb{Z}$, $\varrho(k) \in \mathcal{S}$ and $\varrho(k+1) \in \Pi_\mathcal{V}(\varrho(k))$.*

Definition 3.8. *Let $\mathcal{V}$ denote a combinatorial vector field on a finite cubical complex $\mathcal{X}$. Furthermore assume $\mathcal{S} \subset \mathcal{X}$ to be an invariant set under the flow $\Pi_\mathcal{V}$ and define the exit set of $\mathcal{S}$ by Ex$(\mathcal{S}) := Cl\ \mathcal{S} \backslash \mathcal{S}$.*

Then the invariant set $\mathcal{S}$ is called an isolated invariant set, if the following two statements hold:

(i) The exit set Ex$(\mathcal{S})$ is closed in the combinatorial sense.

(ii) There exists no solution $\varrho : [-1,1] \cap \mathbb{Z} \to \mathcal{X}$ of the flow $\Pi_\mathcal{V}$ such that both $\varrho(-1) \in \mathcal{S}$ and $\varrho(1) \in \mathcal{S}$ hold as well as $\varrho(0) \in$ Ex$(\mathcal{S})$.

One can easily construct combinatorial examples for which one of the two conditions in Definition 3.8 fails and that creates internal tangency of trajectories with the invariant neighborhood in background time continuous dynamic which interrupts the neighborhood isolation.

In applications, as well as proofs, the second condition in Definition 3.8 is not very straight forward to verify. Thus, we have the following equivalent statement for isolation which is more tangible.

Lemma 3.9. *Let $\mathcal{V}$ denote a combinatorial vector field on a finite cubical complex $\mathcal{X}$, and $\mathcal{S}$ be an isolated invariant set under the flow $\Pi_\mathcal{V}$. Then for every cube $\sigma \in \mathcal{X}$, $\sigma^- \in \mathcal{S}$ if and only if $\sigma^+ \in \mathcal{S}$. In other words, $\sigma^\pm$ either both lie in or outside of $\mathcal{S}$.*

Proof. Assume $\mathcal{S}$ is an isolated invariant set. Assume $\sigma^- \in \mathcal{S}$ and $\sigma^+ \notin \mathcal{S}$. By definition of $\mathcal{V}$ and σ^+, we have $\sigma^+ \in \operatorname{Ex}(\mathcal{S})$. on the other hand $\sigma^- \in cl(\sigma^+)$. all together means $\operatorname{Ex}(\mathcal{S})$ is not closed as $\sigma^+ \in \operatorname{Ex}(\mathcal{S})$ but $\sigma^- \in cl(\sigma^+)$ and $\sigma^- \notin \operatorname{Ex}(\mathcal{S})$.

Now assume $\sigma^+ \in \mathcal{S}$ but $\sigma^- \notin \mathcal{S}$. Clearly $\sigma \notin \operatorname{Fix}(\mathcal{V})$. Since $\mathcal{S}$ is invariant, there is a full solution ϱ under $\Pi_{\mathcal{V}}$ within $\mathcal{S}$ which passes through σ^+. But by definition of the flow $\Pi_{\mathcal{V}}$ for a cube in $\operatorname{Dom}(\mathcal{V}) \backslash \operatorname{Fix}(\mathcal{V})$, and definition of a solution, we must have $\sigma^- \in \mathcal{S}$, otherwise that ϱ can not pass through σ^+ and stay within $\mathcal{S}$. $\square$

Proposition 3.10. *Under the assumptions of Definition* 3.8*, an invariant set* $\mathcal{S}$ *is isolated, if the following two statements hold:*

(i) The exit set Ex$(\mathcal{S})$ *is closed in the combinatorial sense,*

(ii) for every $\sigma \in \mathcal{X}$, $\sigma^{\pm}$ *either both lie in or outside of* $\mathcal{S}$.

Proof. Assume the invariant set $\mathcal{S}$ is isolated. Then by Definition 3.8, $\operatorname{Ex}(\mathcal{S})$ is closed. Moreover, Lemma 3.9 implied the second assertion holds.

Conversely, assume the exit set of $\mathcal{S}$, $\operatorname{Ex}(\mathcal{S})$ is closed and for every $\sigma \in \mathcal{X}$, $\sigma^{\pm}$ either both lie in or outside of $\mathcal{S}$. To prove S is isolated invariant set, it remains to verify that the second statement in Definition 3.8 is valid. Assume it is not, then look for a contradiction. So let $\varrho : [-1, 1] \cap \mathbb{Z} \to \mathcal{X}$ be a solution of the flow $\Pi_{\mathcal{V}}$ such that both $\varrho(-1) \in \mathcal{S}$ and $\varrho(1) \in \mathcal{S}$ hold as well as $\varrho(0) \in \operatorname{Ex}(\mathcal{S})$. If $\varrho(0) \in \operatorname{Dom}(\mathcal{V})$, then we get a contradiction as we get $\varrho(0)^+ = \varrho(1)$ with $\varrho(1)^- \notin \mathcal{S}$ while $\varrho(1)^+ \in \mathcal{S}$. If $\varrho(0) \in \operatorname{Im}(\mathcal{V})$ then, since $\varrho(0) \notin \operatorname{Fix}(\mathcal{V})$, so $\varrho(-1) \in \operatorname{Dom}(\mathcal{V})$ which is contradiction by the similar reason argued in previous case. Now assume $\varrho(0) \in \operatorname{Fix}(\mathcal{V})$. Then $\varrho(1) \in \operatorname{Bd}(\varrho(0))$ but that is against the assumption that $\mathcal{S}$ is closed. $\square$

For $x \in X$, and $\sigma \in \mathcal{X}$, there exist relations among λ-characteristic set, $\mathcal{X}^\lambda(x)$, the minimal, maximal cubes of x, $c_m(x)$, $c_M(x)$ resp. and the closure of λ- cell of σ, $\operatorname{cl}\langle\sigma\rangle_\lambda$. These relations will be very helpful in future proofs in this chapter. They are stated and verified next.

Lemma 3.11. *Let* $\mathcal{X}$ *be a cubical complex in* $\mathbb{R}^d$*. Moreover, let* $\sigma \in \mathcal{X}$ *and* $x \in X$*. Then, for* $0 < \lambda < \frac{1}{2}$ *the following statements are equivalent:*

(i) $\sigma \in \mathcal{X}^\lambda(x)$,

(ii) $c_m^\lambda(x) \subset \sigma \subset c_M^\lambda(x)$,

(iii) $x \in cl\big\langle\sigma\big\rangle_\lambda$.

Proof. We recall that

$$\mathcal{X}^\lambda(x) = \{\sigma \in \mathcal{X} \mid \forall v \in \sigma,\ d_v(x) \leq 1-\lambda;\ \forall v \notin \sigma,\ d_v(x) \geq 1-\lambda\}$$

and

$$\mathrm{cl}\big\langle\sigma\big\rangle_\lambda = \{x \in X \mid \forall v \in \sigma,\ d_v(x) \leq 1-\lambda;\ \forall v \notin \sigma,\ d_v(x) \geq 1-\lambda\}$$

hence $\sigma \in \mathcal{X}^\lambda(x)$ iff $x \in \mathrm{cl}\big\langle\sigma\big\rangle_\lambda$ so (i) and (iii) are equivalent. Moreover,

$$c_m^\lambda(x) := \big\{v \in \mathcal{X}_0 \mid d_v(x) < 1-\lambda\big\}$$

and

$$c_M^\lambda(x) := \big\{v \in \mathcal{X}_0 \mid d_v(x) \leq 1-\lambda\big\}.$$

Hence $\sigma \in \mathcal{X}^\lambda(x)$ then for every $v \in \sigma$, $d_v(x) \leq 1-\lambda$ and for every $v \notin \sigma$, $d_v(x) \geq 1-\lambda$ hence all $v \in \mathcal{X}_0$ for which $d_v(x) < 1-\lambda$ must belong to σ hence $c_m^\lambda(x) \subseteq \sigma$. On the other hand for every $v \in \sigma$, $d_v(x) \leq 1-\lambda$ hence $v \in c_M^\lambda(x)$, that is $\sigma \subset c_M^\lambda(x)$. Conversely, if $c_m^\lambda(x) \subset \sigma \subset c_M^\lambda(x)$ then for every $v \in \sigma$, $d_v(x) < 1-\lambda$ and for every $v \notin \sigma$, $v \notin c_m^\lambda(x)$ that is $d_v(x) \geq 1-\lambda$. All together we conclude $\sigma \in \mathcal{X}^\lambda(x)$. Therefore (i) and (ii) are equivalent.

□

Lemma 3.12. *Let $\mathcal{V}$ denote a combinatorial vector field on a finite cubical complex $\mathcal{X}$ in the sense of Definition* 3.5 *and assume $0 < \lambda < \frac{1}{2}$. Then, for every cube $\sigma \in \mathcal{X}\backslash Fix(\mathcal{V})$, there exists a continuous homotopy $h : \sigma^+ \times [0,1] \to \sigma^+$ with following properties:*

(1) The map $h(.,0)$ is identity map σ^+,

(2) For every $s \in [0,1]$ the mapping $h(.,s)$ is the identity map on $|Bd(\sigma^+)|$,

(3) For all $s \in [0,1]$ the inclusion $h(A_\sigma, s) \subset A_\sigma$ holds, and $h(x,1) \in \sigma^-$ for arbitrary point $x \in A_\sigma$,

(4) The image of B_σ under $h(.,1)$ is σ^+,

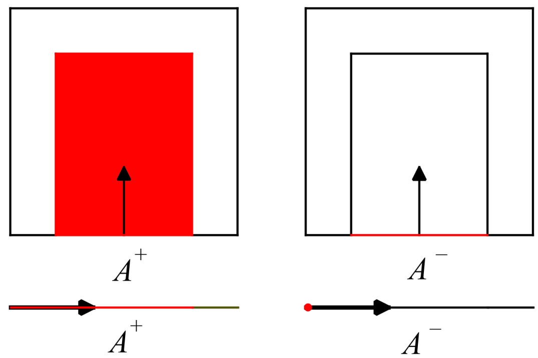

Figure 3.9: the sets A^+ and A^- for a cube of dimension 1 and 2 are shown in red color.

(5) The image of C_σ under $h(.,1)$ is σ^-,

Consequently, the sets A_σ, B_σ, and C_σ are contractible.

Proof. The cube σ^- is facet of σ^+. Assume $\sigma^- = [v_1, ..., v_{2^{k-1}}]$ and let $v_{2^{k-1}+1}, \cdots, v_{2^k}$ be the remaining 2^{k-1} vertices of σ^+. The cube σ^+ can be viewed as $\sigma^- \star \{v_{2^{k-1}+1}, \cdots, v_{2^k}\}$. The operation $\star$ between two cubes σ_1 and σ_2 of same dimension k, produces a cube of dimension $k+1$ which contains σ_1 and σ_2 as opposite faces. Define

$$A^+ := \{x \in \sigma^+ \mid \forall v \in \sigma^-, d_v(x) \leq 1 - \lambda\}, \tag{3.32}$$
$$A^- := \sigma^- \cap \mathrm{cl}\langle \sigma^- \rangle_\lambda. \tag{3.33}$$

In Figure 3.9, the sets A^+ and A^- for dimension 1 and 2 are shown.

In fact, the set $A^+ = A^- \star \{y_{2^{k-1}+1}, \cdots, y_{2^k}\}$ is a cube with base A^- where $\{y_{2^{k-1}+1}, \cdots, y_{2^k}\}$ are the vertices of A^-. Clearly, $A_\sigma = \sigma^- \cup A^+$. Moreover, $A^- \subset \sigma^-$ is a cube of the same dimension as σ^- that is $k-1$. Hence, $A^+ \subset \sigma^+$ has the same dimension as σ^+. Moreover, $C_\sigma = (\sigma^- \backslash \langle \sigma^- \rangle_\lambda) \cup C^+$ where

$$C^+ := \mathrm{bd}(A^+) \cap \mathring{\sigma}^+ \tag{3.34}$$

The homotopy h is constructed in two steps. The first homotopy denoted by h_1 collapses the cube $[y_{2^{k-1}+1}, \cdots, y_{2^k}]$ into a point by deforming $\mathrm{int}(\sigma^+)$ while

h_1 restricted to $\mathrm{bd}(\sigma^+)$ is identity. The second homotopy h_2 is constructed along the intervals $\{z\} \star \{z'\}$ where z' is the orthogonal projection of z on the cube $[x_{2^{k-1}+1}, \cdots, x_{2^k}]$, for every $z \in \sigma^-$. Obviously, for every $z \in \sigma^- \backslash \mathrm{cl}\langle \sigma^- \rangle$, $(\{z\} \star \{z'\}) \cap A_\sigma = \{z\}$. On the other hand for $z \in A^-$, $(\{z\} \star \{z'\}) \cap A_\sigma = \{z\} \star \{x_z\}$ for a unique $x_z \in C^+$. In former case, the homotopy h_2 is defined as identity. In second case, the homotopy h_2 expands the interval $\{x_z\} \star \{z'\}$ to $\{z\} \star \{z'\}$ and contracts the interval $\{x_z\} \star \{z\}$ to $\{z\}$. Concatenation of homotopies h_1 and h_2 produces the desired homotopy.

The properties $(1), (2), (3)$ imply that σ^- is a deformation retraction of A_σ which means A_σ is contractible due to contractibility of σ^-. By (4) and (5), we conclude that B_σ and C_σ are contractible respectively.

$\square$

A visual construction of the homotopy h in Lemma 3.12, on a cube of dimension 2 is given next.

The behavior of h_1 in a few discrete steps is shown in Figure 3.10. In Figure 3.10 the first row from left to right, the top edge of A_σ the red cube shrinks horizontally to a point, while the boundary of the cube shown in black always remains fix. In the second row, from left to right the top face of A_σ shrinks from four sides to a point while the topological boundary point of the cube shown in black always remains fix.

The behavior of h_2 in a few discrete steps is shown in Figure 3.11. In Figure 3.11, the first row, shows how the homotopy h_2 collapses the red tent shape to its base while the topological boundary of the cube remains fix. The homotopy h_2 is combination of vertical shrinks and stretches. Homotopy h_2 applied to h_1 of a cube of dimension 3, is shown in second row of Figure 3.11. The pyramid shape will collapse to its base. Through this homotopy the topological boundary of the cube always remains fix.

The rest of this chapter is devoted to prove that the multi-valued map F defined in Equation 3.30 is upper semi-continuous, and it has non-empty, compact and acyclic image. Before the main theorem, there is a series of preliminaries to state and prove.

We recall that for every $x \in X$, the set $F(x)$ defined in Equation 3.31 was given by

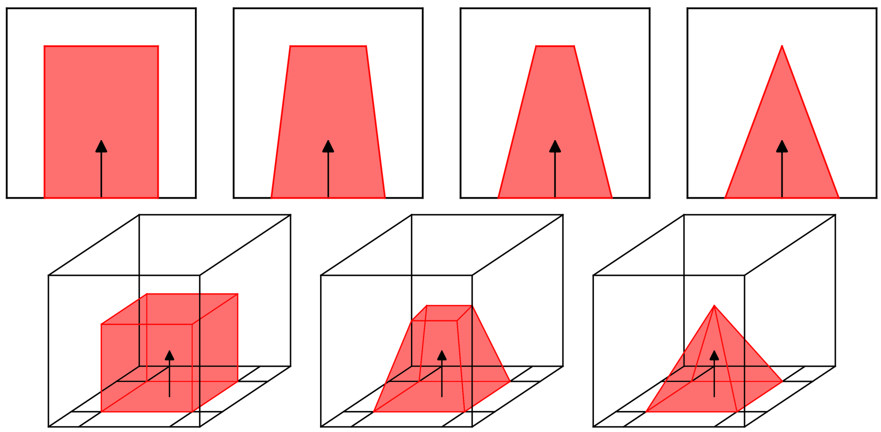

Figure 3.10: In the first row, the homotopy h_1 on a cube of dimension 2 and in the second row the homotopy h_1 on a cube of dimension 3 is shown.

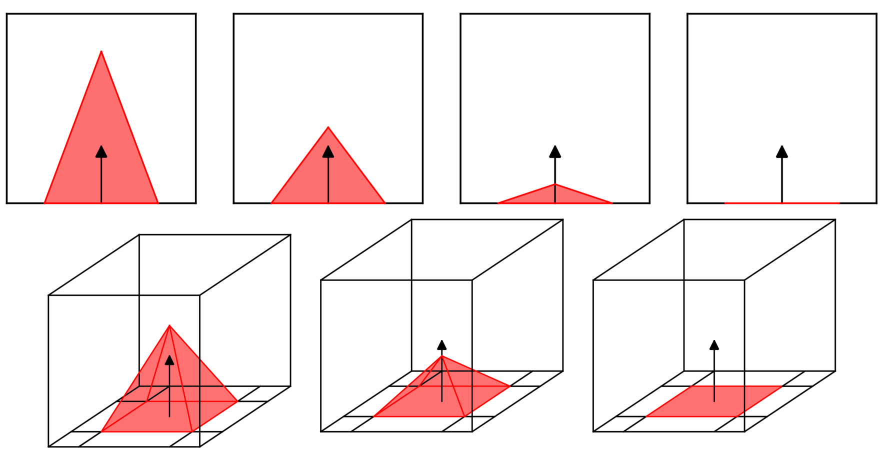

Figure 3.11: In the first row, the homotopy h_2 on a cube of dimension 2 and in the second row the homotopy h_2 on a cube of dimension 3 is shown.

$$F(x) := \bigcup_{\sigma \in \mathcal{X}} F_\sigma(x). \tag{3.35}$$

In Equation 3.35, the union contains some terms which can be eliminated. That can make the union over a smaller set of cubes that reduces the different cases to consider in the proofs related to F . This is more precisely stated and proved next.

Lemma 3.13. *Let $\mathcal{X}$ be a finite cubical complex, and $\mathcal{V}$ be a combinatorial vector field on $\mathcal{X}$. For $\sigma \in \mathcal{X}$, if $\sigma \in \mathcal{X}^\lambda(x) \backslash \{c_M(x)\}$ such that*

(i) $\sigma \neq \sigma^-$, or

(ii) $\sigma = \sigma^-$, and $\sigma^+ \in Cl(c_M(x))$

then, $F_\sigma(x) \subset F_{c_M(x)}(x)$.

Proof. Let $\sigma \in \mathcal{X}$. For simplicity, denote $c_M(x)$ by τ.

(i) Assume $\sigma \neq \sigma^-$, then we have $\sigma = \sigma^+$. By definition of F_σ also $\sigma \in \mathcal{X}^\lambda(x)$, we have $F_\sigma(x) \subset \sigma^+ = \sigma \subset \tau$. Now we have the following possibilities:

   (a) $\tau \in \mathrm{Fix}(\mathcal{V})$. Then $F_\tau(x) = \tau$. Therefore, $F_\sigma(x) \subset F_\tau(x)$.

   (b) $\tau \in \mathrm{Dom}(\mathcal{V}) \backslash \mathrm{Fix}(\mathcal{V})$ then, $\tau = \tau^- \neq \tau^+$. therefore, $F_\tau(x) = C_\tau$. But $\sigma \subsetneq \tau$ because $\tau = \tau^- \neq \tau^+$, and $\sigma = \sigma^+ \neq \sigma^-$. So $\sigma \in \mathrm{Bd}(\tau)$. but $\mathrm{Bd}(\tau^-) \subset C_\tau$ that is $\mathrm{Bd}(\tau) \subset C_\tau$ hence, $\sigma \subset C_\tau$, that means $F_\sigma(x) \subset F_\tau(x)$.

   (c) $\tau \in \mathrm{Im}(\mathcal{V}) \backslash \mathrm{Fix}(\mathcal{V})$, hence, $\tau = \tau^+ \neq \tau^-$ therefore $F_\tau(x) = B_\tau$. Since σ is a proper face of τ so $\sigma \subset \mathrm{Bd}(\tau)$. Also we have $\sigma \neq \tau^-$ because if so, then σ is both source and target which requires σ to be a fixed cube that is not the case. Therefore, $\sigma \subset \mathrm{Bd}(\tau)$ and $\sigma \neq \sigma^-$ implies that $\sigma \subset B_\tau$ which gives $F_\sigma(x) \subset F_\tau(x)$.

(ii) Now we assume, $\sigma \in \mathcal{X}^\lambda(x) \backslash \{c_M(x)\}$ such that $\sigma = \sigma^-$ and $\sigma^+ \in \mathrm{Cl}(c_M(x))$. We have

$$F_\sigma(x) \subset \sigma^+ \subset |\mathrm{Cl}(\tau)| \subset \tau.$$

Since $\sigma = \sigma^+$ takes us back to previous case which is already discussed, hence we may assume $\sigma \neq \sigma^+$. Now we consider the following cases:

   (a) $\tau \in \mathrm{Fix}(\mathcal{V})$. Then $F_\tau(x) = \tau$. Therefore, $F_\sigma(x) \subset F_\tau(x)$.

(b) $\tau \in \mathrm{Dom}(\mathcal{V})\backslash\mathrm{Fix}(\mathcal{V})$ then, $\tau = \tau^- \neq \tau^+$. therefore, $F_\tau(x) = C_\tau$. Since σ^+ is not a fixed target and τ is not a fixed source so $\sigma^+ \neq \tau$. Hence, $\sigma^+ \in \mathrm{Cl}(\tau)\backslash\{\tau\}$ that is $\sigma^+ \in \mathrm{Bd}(\tau)$. But since $\tau = \tau^-$ so $\mathrm{Bd}(\tau) \subset C_\tau$. All together we have $\sigma^+ \subset C_\tau$ which implies $F_\sigma(x) \subset F_\tau(x)$.

(c) Finally, assume $\tau \in \mathrm{Im}(\mathcal{V})\backslash\mathrm{Fix}(\mathcal{V})$, hence, $\tau = \tau^+ \neq \tau^-$ therefore $F_\tau(x) = B_\tau$. If $\sigma^+ = \tau$ then since $\tau = \tau^+$ thus, $\sigma^- = \tau^-$. So we have $\sigma = \sigma^- = \tau^-$ also by our assumption $\sigma \neq \sigma^+$ therefore $F_\sigma(x) = C_\sigma$. But $C_\sigma = C_{\tau^-} = C_\tau \subset B_\sigma$, that is $F_\sigma(x) \subset F_\tau(x)$. Now consider $\sigma^+ \neq \tau = \tau^+$ thus $\sigma^- \neq \tau^-$ that is $\sigma \neq \tau^-$. Also, by assumption $\sigma^+ \in \mathrm{Cl}(\tau)$ we have $\sigma^+ \in \mathrm{Bd}(\tau)$. This yields

$$F_\sigma(x) \subset \sigma^+ \subset |\mathrm{Bd}(\tau)\backslash\{\tau^-\}| \subset B_\tau = F_\tau(x). \tag{3.36}$$

□

Corollary 3.14. *Assume $\mathcal{X}$ is a finite cubical complex. Moreover, let $\mathcal{V}$ be a combinatorial vector field on $\mathcal{X}$ and F be the multi-valued map associated with it defined in Equation* 3.31*. Then, for every $x \in X$*

$$F(x) = F_{c_M(x)}(x) \cup \bigcup_{\sigma \in \mathcal{T}^\lambda(x)} F_\sigma(x)$$

where,

$$\mathcal{T}^\lambda(x) = \Big\{\sigma \in \mathcal{X}^\lambda(x)\backslash\{c_M(x)\} \mid \sigma = \sigma^- \text{ and } \sigma^+ \notin \mathit{Cl}\ c_M(x)\Big\}.$$

For every cube $\sigma \in \mathcal{X}$, the map F_σ given in Equation 3.30 is constructed based on the map $\mathcal{X}^\lambda$ and multi-valued map F is defined in terms of F_σ's. The next goal to achieve is to show that F is a strongly upper semi continuous map. First, we show this property holds for $\mathcal{X}^\lambda$ and consequently it holds for F_σ and so it holds for F. This is stated and verified next.

Lemma 3.15. *Let $\mathcal{X}$ denote a finite cubical complex and $0 < \lambda < \frac{1}{2}$. For $x \in X = |\mathcal{X}|$, the set $\mathcal{X}^\lambda(x)$ is non-empty. Moreover, the multi-valued map $\mathcal{X}^\lambda : X \multimap \mathcal{P}(\mathcal{X})$ defined by $x \mapsto \mathcal{X}^\lambda(x)$ is strongly upper semi continuous, i.e. for every $x \in X$, there is a neighborhood U_x of x such that*

$$\forall y \in U_x,\ \mathcal{X}^\lambda(y) \subset \mathcal{X}^\lambda(x). \tag{3.37}$$

Proof. To show $\mathcal{X}^\lambda(x)$ is non-empty set, we show that the minimal cube of x, $c_m(x)$ belongs to $\mathcal{X}^\lambda(x)$. By definition,

$$\mathcal{X}^\lambda(x) = \Big\{\sigma \in \mathcal{X} \mid \forall v \in \sigma,\ d_v(x) \leq 1-\lambda;\ \forall v \notin \sigma,\ d_v(x) \geq 1-\lambda\Big\}$$

and

$$c_m^\lambda(x) = \Big\{v \in \mathcal{X}_0 \mid d_v(x) < 1-\lambda\Big\}.$$

By the choice of λ, $c_m(x)$ contains at least one vertex. Hence $c_m(x)$ is a cube. Then for every $v \in c_m(x)$, $d_v(x) < 1$ so $d_v(x) \leq 1$. Moreover, for every $v \notin c_m(x)$, $d_v(x) \geq 1-\lambda$ otherwise $v \in c_m(x)$. That proves, $c_m(x) \in \mathcal{X}^\lambda(x)$. Hence $\mathcal{X}^\lambda(x) \neq \emptyset$.

Now let $x \in X$.

1. If $\mathcal{X}^\lambda(x)$ has only one cube, then we have $c_m(x) = c_M(x)$. In this case, $x \in \text{Int } c_M(x)$. Now let U_x be a neighborhood of x with $U_x \subset \text{Int } c_M(x)$. Then for all $y \in U_x$, $\mathcal{X}^\lambda(y) = c_M(x) = \mathcal{X}^\varepsilon(x)$

2. If $\mathcal{X}^\lambda(x)$ has more than one cube. This case happens when x is located on the boundary of a λ-cell. Then by finitness of cubical complex $\mathcal{X}$, x is located on boundary of finitely many λ cells $L_1, \cdots, L_r$. There exists neighborhood U_x of x such that $U_x \subset \bigcup_{i=1}^r L_i$. Then for every $y \in U_x$, $\mathcal{X}^\lambda(y) \subset \{\sigma_1, \cdots, \sigma_r\}$ where σ_i is the associated cube to λ-cell L_i for $i = 1, \cdots, r$. Therefore, $\mathcal{X}^\lambda(y) \subset \{\sigma_1, \cdots, \sigma_r\} = \mathcal{X}^\lambda(x)$.

$\square$

A more intuitive proof for Lemma 3.15 *is given next.*

Proof. By contrary assume, $\mathcal{X}^\lambda$ is not strongly upper semi-continuous. Therefore, for every neighborhood U of x there is $y \in U$ such that $\mathcal{X}^\lambda(y) \not\subset \mathcal{X}^\lambda(x)$. Therefore, one can find a sequence $\{y_n\}_n$ of points and a sequence $\{\sigma_n\}_n$ of cubes such that $\sigma_n \in \mathcal{X}^\lambda(y_n) \backslash \mathcal{X}^\lambda(x)$, associated with the open neighborhoods of x, $U_n = B(x, \frac{1}{2^n})$. Hence the sequence y_n converges to x and by finiteness of $\mathcal{X}$, $\{\sigma_n\}_n$ has a constant sub-sequence $\{\sigma_{n_i}\}_{n_i}$ where $\sigma_{n_i} = \sigma$, for some $\sigma \in \mathcal{X}^\lambda$. We still have $\{y_{n_i}\}_{n_i}$ is a convergent sequence to x. By definition of $\mathcal{X}^\lambda(y_{n_i})$, for every $v \in \sigma$, $d_v(y_{n_i}) \leq 1-\lambda$ and for every $v \notin \sigma$, $d_v(y_{n_i}) \geq 1-\lambda$. Now by continuity of function d_v and convergence of $\{y_{n_i}\}_{n_i}$, we have for every $v \in \sigma$, $d_v(x) \leq 1 - \lambda$ and for every $v \notin \sigma$, $d_v(x) \geq 1 - \lambda$. That means $\sigma \in \mathcal{X}^\lambda(x)$ which is contraction with the construction of sequence $\{\sigma_n\}_n$. This contradiction implies $\mathcal{X}^\lambda$ is strongly upper semi-continuous.

□

For every $\sigma \in \mathcal{X}$, the multivalued map F_σ defined in Equation 3.30 and consequently the multivalued map F given in Equation 3.31 inherit the strongly upper semi continuous property from the multivalued map $\mathcal{X}^\lambda$.

Lemma 3.16. *Let $\mathcal{X}$ denote a finite cubical complex and $0 < \lambda < \frac{1}{2}$. For every $\sigma \in \mathcal{X}$, and for $x \in X = |\mathcal{X}|$, there is a neighborhood U_x of x such that,*

$$\forall y \in U_x, \quad F_\sigma(y) \subset F_\sigma(x). \tag{3.38}$$

In other words, for every $\sigma \in \mathcal{X}$, the multivalued map F_σ is strongly upper semi continuous.

Proof. For every $x \in X$ by Lemma 3.15, there exists a neighborhood U_x of x such that

$$\forall y \in U_x, \ \mathcal{X}^\lambda(y) \subset \mathcal{X}^\lambda(x). \tag{3.39}$$

Therefore, $c_M(y) \subseteq c_M(x)$. Moreover, for every $\sigma \in \mathcal{X}^\lambda(x) \backslash \mathcal{X}^\lambda(y)$,

$$F_\sigma(y) = \emptyset \subseteq F_\sigma(x).$$

So let $\sigma \in \mathcal{X}^\lambda(y)$. Following cases are possible:

1. $F_\sigma(x) = \emptyset$. In this case we should have $\sigma \notin \mathcal{X}^\lambda(x)$, which is not possible by choice of σ.

2. $F_\sigma(x) = \sigma$ then $\sigma = c_M(x)^+ = c_M(x)^-$. By definition of F_σ, $F_\sigma(y) \subseteq \sigma^+ = \sigma = F_\sigma(x)$.

3. $F_\sigma(x) = C_\sigma$ which happens when $\sigma = c_M(x)^-$ and $\sigma \neq c_M(x)^+$. Then, $F_\sigma(y) \in \{\emptyset, C_\sigma\}$, then $F_\sigma(y) \subseteq C_\sigma = F_\sigma(x)$. Note that $F_\sigma(y) \notin \{\sigma, B_\sigma\}$ as $\sigma \in \text{dom}(\mathcal{V}) \backslash \text{Fix}(\mathcal{V})$. Moreover, $F_\sigma(y) \neq A_\sigma$ otherwise, we require to have $\sigma \neq c_M(y)^-$ and $\sigma \neq c_M(y)^+$. while if $c_M(x)^+ \notin \mathcal{X}^\lambda(x)$ then $c_M(x)^- = c_M(x)$. so $\sigma = c_M(x)$. But $\sigma \in \mathcal{X}^\lambda(y) \subseteq \mathcal{X}^\lambda(x)$. That implies $\mathcal{X}^\lambda(y) = \mathcal{X}^\lambda(x)$ thus $\sigma = c_M(x)^- = c_M(y)^-$. Now if $c_M(x)^+ \in \mathcal{X}^\lambda(x)$

then $c_M(x) = c_M(x)^+$. as $\sigma = c_M(x)^- \neq c_M(x)^+ = c_M(x)$ and $\sigma = c_M(x)^- \subseteq c_M(y) \subseteq c_M(x) = c_M(x)^+$ then either $c_M(y) = c_M(x)^-$ or $c_M(y) = c_M(x)$, which implies $\sigma = c_M(y)^-$. This completes the claim of $F_\sigma(y) \neq A_\sigma$.

4. $F_\sigma(x) = A_\sigma$ which happens when $\sigma \in \mathcal{X}^\lambda(x)$ and $\sigma \neq c_M(x)^+$ and $\sigma \neq c_M(x)^-$. Then, $F_\sigma(y) \in \{\emptyset, \sigma, C_\sigma, A_\sigma\}$, thus $F_\sigma(y) \subseteq A_\sigma = F_\sigma(x)$. Note that $F_\sigma(y) = B_\sigma$ is not possible as if so then $\sigma = c_M(y)^+$ and $\sigma \neq c_M(y)^-$. So, $c_M(y)^- \subsetneq c_M(y)^+ = \sigma \subsetneq c_M(x)$, hence σ is a proper face of $c_M(x)$. But $A_\sigma \subseteq \sigma^+$. Since $\sigma = c_M(y)^+$ hence $\sigma = \sigma^+$. Thus, $A_\sigma \subseteq \sigma^+ = \sigma$ which is possible only if σ is a fixed cube which is not the case here. This contradiction implies $F_\sigma(y) \neq B_\sigma$.

5. $F_\sigma(x) = B_\sigma$ which happens when $\sigma = c_M(x)^+$ and $\sigma \neq c_M(x)^-$. Then, $F_\sigma(y) \in \{\emptyset, C_\sigma, B_\sigma\}$, thus, $F_\sigma(y) \subseteq C_\sigma = F_\sigma(x)$. Note that $F_\sigma(y) \in \{\sigma, A_\sigma\}$ is not possible. If $F_\sigma(y) = \sigma$ then $\sigma = c_M(y)^+ = c_M(y)^+$ which implies $\sigma \in \text{Fix}(\mathcal{V})$ that is contradiction. Now if, $F_\sigma(y) = A_\sigma$ then, $\sigma \neq c_M(y)^-$ and $\sigma \neq c_M(y)^+$. But $\sigma \in \mathcal{X}^\lambda(y) \subseteq \mathcal{X}^\lambda(x)$ and $\sigma = c_M(x)^+$ so $c_M(x)^+ \in \mathcal{X}^\lambda(x)$ hence $\sigma = c_M(x)^+ = c_M(x) = c_M(y)$ but this is in contrast with $\sigma \neq c_M(y)^-$ and $\sigma \neq c_M(y)^+$.

$\square$

Assume X and Y are arbitrary sets and $f : X \to Y$ is a single valued function. For subset $B \subset Y$, the pre-image of B under f is defined to be

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}. \tag{3.40}$$

Now assume $F : X \multimap Y$ is a multi-valued map and $B \subset Y$. Since for every $x \in X$, $F(x) \subset Y$ then the concept of pre-image under a multi-valued map is not immediately clear. Two versions of pre-image for a multi-valued map are defined in [1], namely *small pre-image* and *large pre-image* given respectively by

$$F^{-1}(B) = \{x \in X \mid F(x) \subset B\}, \tag{3.41}$$

$$F^{*-1}(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}. \tag{3.42}$$

A single valued map $f : X \to Y$ can be viewed as a particular case of a multi-valued map by defining $x \to \{f(x)\}$. In this case the concepts, small pre-image and large pre-image coincide.

Assume X and Y are arbitrary topological spaces. A function $f : X \to Y$ is called continuous if and only if for every $B \subset Y$ open in Y, $f^{-1}(B)$ is open in X. For a multi-valued map $F : X \to Y$, depending on which notion is used for pre-image, one obtains the following two versions of continuity.

Definition 3.17. *Assume X and Y be topological spaces and let $F : X \multimap Y$ be a multi-valued map. Then F is called upper semi-continuous if for every open $B \subset Y$, the small pre-image $F^{-1}(B)$ is open in X. The multi-valued map $F : X \multimap Y$ is called lower semi-continuous if for every open $B \subset Y$, the large pre-image $F^{*-1}(B)$ is open in X.*

The concepts of upper and lower semi-continuous can be also defined in terms of closed sets. The Multi-valued map $F : X \multimap Y$ is called *upper semi-continuous* if for every closed set $C \subset Y$, the large pre-image $F^{*-1}(C)$ is closed in X, and it is called *lower semi-continuous* if for every closed set $C \subset Y$, the small pre-image $F^{-1}(C)$ is closed in X.

Smaller classes of maps between two topological spaces X and Y are called *strongly upper semi-continuous* and *strongly lower semi-continuous* are defined as follows.

Definition 3.18. *Assume X and Y be topological spaces and let $F : X \multimap Y$ be a multi-valued map. Then F is called strongly upper semi-continuous if for every subset $B \subset Y$, the small pre-image $F^{-1}(B)$ is open in X. The multi-valued map $F : X \multimap Y$ is called strongly lower semi-continuous if for every subset $B \subset Y$, the large pre-image $F^{*-1}(B)$ is open in X.*

The main theorem of this chapter comes next. It states the designated properties of multi-valued maps F which are required in further development of the Conley index theory for cubical complexes.

Theorem 3.19. *Let $\mathcal{X}$ denote a finite cubical complex and let $\mathcal{V}$ be a combinatorial vector field on $\mathcal{X}$ in the sense of Definition* 3.5*, and assume that $0 < \varepsilon < \lambda < \frac{1}{2}$ holds. Furthermore, let $F : X \multimap X$ be defined as in Definition* 3.31*. Then, F is strongly upper semi-continuous, and its values $F(x)$ for all $x \in X$ are non-empty and contractible sets.*

Proof. Let $X = |\mathcal{X}|$ that is geometric realization of $\mathcal{X}$. For all $x \in X$, $\mathcal{X}^\varepsilon(x) \neq \emptyset$, and the map $\mathcal{X}^\varepsilon$ is strongly upper semi- continuous. Consequently, for every $x \in X$, $F(x)$ is non-empty and by Lemma 3.16 the map F is strongly upper semi-continuous .

For every $x \in X$, there is neighborhood of x, U_x such that for every $y \in U_x$, $\mathcal{X}^\varepsilon(y) \subset \mathcal{X}^\varepsilon(x)$. For the same neighborhood U_x of x, then,

$$F(y) = \bigcup_{\sigma \in \mathcal{X}^\varepsilon(y)} F_\sigma(y) \subset \bigcup_{\sigma \in \mathcal{X}^\varepsilon(x)} F_\sigma(x) = F(x) \tag{3.43}$$

Note that $F_\sigma(y) \subset F_\sigma(x)$ for all $y \in U_x$ as showed in Lemma 3.16.

Now we show that image of F is contractible. Since every cube is contractible and for every $\sigma \in \mathcal{X}$ and $x \in X$, $F_\sigma(x)$ is either empty or it is contractible as it is homotopic to cube σ^+. Now, to prove $F(x)$ is contractible we should prove $\bigcup_{\sigma \in \mathcal{X}^\varepsilon(x)} F_\sigma(x)$ is contractible. In general, the union of contractible sets is not necessarily contractible. For contractibility of $F(x)$, the representation

$$F(x) = F_{c_M(x)}(x) \cup \bigcup_{\sigma \in \mathcal{T}^\varepsilon(x)} F_\sigma(x) \tag{3.44}$$

where,

$$\mathcal{T}^\varepsilon(x) = \Big\{ \sigma \in \mathcal{X}^\varepsilon(x) \backslash \{c_M(x)\} \mid \sigma = \sigma^- \text{ and } \sigma^+ \notin \text{Cl } c_M(x) \Big\} \tag{3.45}$$

will come in handy.

Let $x \in X$ be fixed until the end of the proof and denote $c_M(x)$ by τ. If the set $\mathcal{T}^\varepsilon(x) = \emptyset$, then $F(x) = F_\tau(x)$ which is contractible by Lemma 3.12. Therefore, suppose that $\mathcal{T}^\varepsilon(x) \neq \emptyset$. We show $F_\tau(x) \cup F_\sigma(x)$, for every $\sigma \in \mathcal{T}^\varepsilon(x)$ is contractible.

1. If $\tau \in \text{Fix}\mathcal{V}$ then $\tau = \tau^- = \tau^+$, and by definition of F_τ we have $F_\tau(x) = \tau$. By choice of σ, $\sigma = \sigma^-$ and $\sigma^+ \notin \text{Cl}\tau$. Therefore, σ can neither be equal to τ^+ nor equal to τ^-. In this case, definition of F_σ gives $F_\sigma(x) = A_\sigma$ and so $F_\tau(x) \cup F_\sigma(x) = \tau \cup A_\sigma$. But $A_\sigma \subset \sigma^+$ and $\sigma^+ \notin \text{Cl}\tau$, hence $A_\sigma \cap \tau = \sigma$ which is a convex set. Since, τ and A_σ are contractible, and their intersection is contractible, thus $\tau \cup A_\sigma$ is contractible. Since $\tau \cap A_\sigma = \sigma^-$

and A_σ is homotopy equivalent to σ^-, with homotopy being identity on the boundary of σ^+, therefore $\tau \cup A_\sigma$ is homotopy equivalent to τ.

2. Now, assume $\tau \in \text{dom}(\mathcal{V})\backslash\text{Fix}(\mathcal{V})$ and let $\sigma \in \mathcal{T}^\varepsilon(x)$. In this case, $\tau = c_M(x)^-$ and $\tau \neq c_M(x)^+$. Therefore, by definition of multi-valued map F_τ, $F_\tau(x) = C_\tau$. By choice of σ, it's a proper face of $c_M(x) = \tau = \tau^- \subsetneq \tau^+$. Hence $\sigma \neq c_M(x)^-$, and $\sigma \neq c_M(x)^+$. By definition of F_σ, we have $F_\sigma(x) = A_\sigma$. Thus, $F_\tau(x) \cup F_\sigma(x) = C_\tau \cup A_\sigma$. On the other hand, $C_\tau \cap A_\sigma = \sigma$ as C_τ contains boundary of $\tau^- = \tau$, $\sigma = \sigma^- \subset \tau$, $A_\sigma \subset \sigma^+$ and $\sigma^+ \notin$ Cl τ. Since C_τ and A_σ are contractible as they deform into τ^- and σ^+ respectively, and $C_\tau \cap A_\sigma$ is contractible, so $F_\tau(x) \cup F_\sigma(x) = C_\tau \cup A_\sigma$ is contractible. Since $C_\tau \cap A_\sigma = \sigma^-$ and A_σ is homotopy equivalent to σ^- therefore $C_\tau \cup A_\sigma$ is homotopy equivalent to C_τ. But C_τ is homotopy equivalent to τ^-, with homotopy being identity on the boundary of σ^+, therefore $C_\tau \cup A_\sigma$ is homotopy equivalent to $\tau^- = \tau$.

3. Finally, assume $\tau \in \text{im}(\mathcal{V})\backslash\text{Fix}(\mathcal{V})$ and let $\sigma \in \mathcal{T}^\varepsilon(x)$. Then τ^- is a facet of $\tau^+ = \tau$. By definition of F_τ, we have $F_\tau(x) = B_\tau$. On the other hand, $\sigma \neq \tau^-$ otherwise we get $\sigma^+ = \tau^+ = \tau$ which is in contrast with the choice of $\sigma^+ \notin \text{Cl}(\tau)$. Also, $\sigma \subsetneq \tau = \tau^+$, thus definition of F_σ gives, $F_\sigma(x) = A_\sigma$. Now note that, B_τ contains all boundary of τ but the facet τ^-, and the fact that $\sigma \neq \tau^-$ as well as $\sigma \subsetneq \tau$, hence $|\sigma| \subset B_\tau$. Also, $A_\sigma \subset \sigma^+$ and $\sigma^+ \notin \text{Cl}(\tau)$, therefore Moreover, $F_\tau(x) \cap F_\sigma(x) = B_\tau \cap A_\sigma = \sigma$ which is contractible. Since B_τ is homotopy equivalent to $\tau^+ = \tau$, with homotopy being identity on the boundary of τ^+, and A_σ is homotopy equivalent to $\sigma = \sigma^-$, with homotopy being identity on the boundary of σ^+ and $B_\tau \cap A_\sigma = \sigma^- = \sigma$, therefore $B_\tau \cup A_\sigma$ is homotopy equivalent to $\tau^+ = \tau$.

We proved that for every $\sigma \in \mathcal{T}^\varepsilon(x)$, $F_\tau(x) \cup F_\sigma(x)$ is always contractible and it is homotopy equivalent to τ. Now, we should show that the deformations constructed above for different $\sigma \in \mathcal{T}^\varepsilon(x)$ can be concatenated with each other to obtain a deformation which deforms $F(x)$ into τ which implies $F(x)$ is contractible as desired. So assume $\sigma_1, \sigma_2 \in \mathcal{T}^\varepsilon(x)$ be distinct. Then, by properties of elements of $\mathcal{T}^\varepsilon(x)$ and all three possibilities for τ as argued about, we can conclude $F_{\sigma_1}(x) = A_{\sigma_1}$ and $F_{\sigma_2}(x) = A_{\sigma_2}$. By definition, $A_{\sigma_1} \subset {\sigma_1}^+$ and $A_{\sigma_2} \subset {\sigma_2}^+$. Also, ${\sigma_1}^- = \sigma_1 \neq \sigma_2 = {\sigma_2}^-$ implies ${\sigma_1}^+ \neq {\sigma_2}^+$ and we know ${\sigma_1}^+ \notin \text{Cl}(\tau)$ and ${\sigma_2}^+ \notin \text{Cl}(\tau)$. In this case intersection of ${\sigma_1}^+$ and ${\sigma_2}^+$ is a subset of τ. Thus, $A_{\sigma_1} \cap A_{\sigma_2} \subset \text{Bd}(\tau)$. This guarantees that the concatenation of the the homotopies do not conflict. Therefore, $F_\tau(x) \cup F_\sigma$ which is homotopy equivalent to τ which is identity on $\mathrm{Bd}(\tau)$, for $\sigma \in \mathcal{T}^{\varepsilon}(x)$ can be concatenated and have $F(x)$ homotopy equivalent to τ. $\square$

Recently, a new approach toward Conley index theory on simplicial complexes by means of semi-flows is initiated. We invested some effort on initiating an analoguous approach on cubical complexes. The partial results still require to be developed more before publishing. The approach in [13] is left for the future research as it involved considerable amount of proofs. We end this chapter with proposed definitions and conjectures about the Conley index of a cubical complex which are verified to hold for simplicial complexes in [2].

3.5 Index Pair for a Dynamical System in Cubical Setting

The notion of index pair for an isolating neigborhood of an upper semi-continuous map was defined prior to the notion of weak index pair for such maps. In [3] it is shown that even though for any isolating neighborhood the index pair may not exists, the weak index pair always exists. Moreover, the Conley index of a combinatorial isolated invariant set $\mathcal{S}$ is isomorphic to the Conley index defined based on the weak index pair constructed from the isolating neighborhood associated with $\mathcal{S}$.

We propose the following definition for a combinatorial index pair associated with a combinatorial isolated invariant set of a cubical complex.

Assume $\mathcal{X}$ is a finite cubical complex and let $\mathcal{V} : \mathcal{X} \to \mathcal{X}$ be a combinatorial vector field. Moreover, let $\mathcal{S}$ be an isolated invariant set in sense of Definition 3.8. Then, a pair $(\mathcal{P}_1, \mathcal{P}_2)$ of closed subsets of $\mathcal{X}$ such that $\mathcal{P}_2 \subseteq \mathcal{P}_1$ is called an *index pair* for $\mathcal{S}$, if the following three conditions are satisfied:

(i) $\mathcal{P}_1 \cap \Pi_{\mathcal{V}}(\mathcal{P}_2) \subseteq \mathcal{P}_2$,

(ii) $\Pi_{\mathcal{V}}(\mathcal{P}_1 \backslash \mathcal{P}_2) \subseteq \mathcal{P}_1$,

(iii) $\mathcal{S} = \mathrm{Inv}(\mathcal{P}_1 \backslash \mathcal{P}_2)$.

Conjecture 3.20. *The index pair for an isolated invariant set always exists. The pair $(Cl(\mathcal{S}), Ex(\mathcal{S}))$ is an index pair for $\mathcal{S}$.*

Conjecture 3.21. *The (co)homology of an index pair only depends on the isolated invariant set associated with it. Consequently, the Conley index of $\mathcal{S}$ defined as the relative (co)homology of an index pair for $\mathcal{S}$ is well-defined and is denoted by Con(S).*

Assume $X \subset \mathbb{R}^d$ and let $F : X \times \mathbb{Z} \to \mathcal{P}(X)$ be an upper semi-continuous (usc) mapping with compact values. Then, F is called a *discrete multivalued dynamical system* (dmds) if the following statements are satisfied:

(i) $\forall\, x \in X,\ F(x, 0) = \{x\}$,

(ii) $\forall\, m, n \in \mathbb{Z},\ mn \geq 0, \forall\, x \in X,\ F\big(F(x, n), m\big) = F(x, m + n)$,

(iii) $\forall x, y \in X,\ y \in F(x, -1) \Leftrightarrow x \in F(y, 1)$.

Define $F^1 : X \to \mathcal{P}(X)$ to be

$$F^1(x) = F(x, 1). \tag{3.46}$$

The map F^1 is called the generator of the dmds F and it is usually denoted and identifies with F.

A map $\varrho : I \to X$ with I be an interval in $\mathbb{Z}$ is called a solution for F through x if for some $k \in I$, $\varrho(k) = x$ and $\varrho(n+1) \in F(\varrho(n))$ for all $n, n+1 \in \mathrm{dom}\varrho(n)$.

A compact subset of $N \subset X$ is an *isolating neighborhood* of F if

$$\textit{Inv}(N) \subset \textit{int}(N) \tag{3.47}$$

where, *Inv*(N) the *invariant part* of N is defined to be

$$\textit{Inv}(N) = \{x \in N \mid \exists\, \varphi : \mathbb{Z} \to N \text{ a solution under } F \text{ which passes through } x\}. \tag{3.48}$$

There are earlier notions of isolating neighborhood defined in [15] and [27] as *strongly isolating neighborhood* which coincide with Definition 3.47 when F is a single valued map.

A subset $S \subset X$ is called *invariant* under F if Inv$(S) = S$, and it is called an *isolated invariant* if there is an isolating neighborhood N under F such that

$\mathrm{Inv}(N) = S$.

For an isolated invariant subset $S \subset X$ the index pair is defined as follows:

A pair (P_1, P_2) of compact subsets of X such that $P_2 \subset P_1 \subset N$ is called an *index pair* for N, if the following statements hold:

(i) $F(P_i) \cap N \subset P_i$ for $i \in \{1, 2\}$,

(ii) $F(P_1 \backslash P_2) \subset N$,

(iii) $\mathrm{Inv}(N) \subset \mathrm{int}(P_1 \backslash P_2)$,

In application the Conley index pair for multivalued maps are not satisfactory as they engage the concept of isolating neighborhoods which are restrictive. The limitations were removed in [2] by the developed notion of index pair, called *weak index pair*.

A pair (P_1, P_2) of compact subsets of X such that $P_2 \subset P_1 \subset N$ is called a *weak index pair* for N, if the following statements hold:

(i) $F(P_i) \cap N \subset P_i$ for $i \in \{1, 2\}$,

(ii) $\mathrm{cl}(P_1) \cap \mathrm{cl}\big(F(P_1) \backslash P_1\big) \subset P_2$,

(iii) $\mathrm{Inv}(N) \subset \mathrm{int}(P_1 \backslash P_2)$,

(iv) $P_1 \backslash P_2 \subset \mathrm{int}(N)$.

Conjecture 3.22. *Let $\mathcal{V}$ be a combinatorial vector field on a finite cubical complex $\mathcal{X}$. Moreover assume $\mathcal{S} \subset \mathcal{X}$ be an isolated invariant set in the sense of Definition* 3.8. *Let* $0 < \delta < \frac{1}{2}$, *then*

$$N_\delta(\mathcal{S}) = \bigcup_{\sigma \in \mathcal{S}} cl\big\langle \sigma \big\rangle_\delta \tag{3.49}$$

is an isolating block for F. In particular, $N_\delta(\mathcal{S})$ is an isolating neighborhood for F.

Conjecture 3.23. *For $0 < \delta' < \delta$ define*

$$P_1 = N_\delta(\mathcal{S}) \cap N_{\delta'}(\mathcal{S}), \quad \textit{and} \quad P_2 = bd\big(N_\delta(\mathcal{S})\big) \cap N_{\delta'}(\mathcal{S}) \tag{3.50}$$

Then, $P_2 \subset P_1 \subset N_\delta(\mathcal{S})$ are compact and the pair $P = (P_1, P_2)$ in 3.50 is a weak index pair for F and the isolating neighborhood $N_\delta(\mathcal{S})$.

Let $S = \text{Inv}(N_\delta)$. The Conley index of S with respect to F is

$$\text{Con}(S, F) := L(H^*(P_1, P_2), I_P), \tag{3.51}$$

where L is the Leray reduction of the relative cohomology graded module $H * (P_1, P_2)$, and I_P is the index map on $H^*(P_1, P_2)$. In [2] it is proven

$$\text{Con}(S, F) \cong (H^*(P_1, P_2), id_{H^*(P_1,P_2)}), \tag{3.52}$$

where $id_{H^*(P_1,P_2)}$ denotes the identity map.

Conjecture 3.24. *Assume $\mathcal{V}$ is a combinatorial vector field defined on a finite cubical complex $\mathcal{X}$. Denote the multi-valued map associated with $\mathcal{V}$ defined in 3.31 by F.*

$$\textit{Con}(\mathcal{S}) = H^*\big(\textit{Cl}(\mathcal{S}), \textit{Ex}(\mathcal{S})\big) \cong H^*(P_1, P_2) = \textit{Con}(S). \tag{3.53}$$

Conclusion

The work done in this thesis could be counted as the initial steps toward the development of the Conley index theory on combinatorial dynamical system in cubical setting. That theory has been well advanced earlier in the framework of simplicial complexes. However, cubical complexes are more convenient to work with both in the sense of defining the fundamental notions toward Conley index theory as well as convenience in proofs and arguments. Moreover, cubical complexes are convenient for interpretation of data sets as a multi-dimensional cubical grids. Availability of Euclidean metrics and coordinate systems on geometric realization of a cubical complex makes formulations relatively easier than simiplicial complexes. Moreover, in application, computer coding of this theory in cubical setting seems easier. Therefore, computer assisted analysis in Conley index theory could be also implemented on study of finite but probably large dynamical systems in cubical setting.

In Chapter 3 for a given combinatorial vector field $\mathcal{V}$ on a finite cubical complex a combinatorial flow $\Pi_{\mathcal{V}}$ on $\mathcal{X}$ is associated. On the other hand a multi-valued map F on X is associated with $\mathcal{V}$. We prove that F is strongly upper semi continuous with non-empty, compact and acyclic values. Analogous to the simplicial complex setting, the next step is to show a correspondence between solutions under the flow $\Pi_{\mathcal{V}}$ and the solutions of the multi-valued map F, as well as the correspondence between isolated invariant sets with respect to $\Pi_{\mathcal{V}}$ and isolating blocks of F. Conley index for a combinatorial dynamical system as well as for a multi-valued map F are defined. We conjecture that the index pair defined on simplicial complexes could be adapted to cubical setting. The isomorphism between combinatorial Conley index and the Conley index of the associated multi-valued map remains to be proven as a future research direction.

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